FOURIER TRANSFORM AND DISTRIBUTIONAL REPRESENTATIONS WITH SOME APPLICATIONS

BY

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Dedicated to my beloved parents

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ABSTRACT

NAME:	Mohammed H. Al-Lail
TITLE OF STUDY:	Fourier Transform and Distributional Representations with Some Applications
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An important aspect of the analysis of special functions is to find their properties. We use the Fourier transform representation of the generalized hypergeometric functions to evaluate integrals of products of two generalized hypergeometric functions. It turns out that the integral of products of two generalized hypergeometric functions gives generalized Kampé de Fériet's hypergeometric function. A number of integral identities for confluent and Gauss hypergeometric functions are deduced as special cases. We prove that any Mellin transformable function can be represented as a series of Dirac delta functions. This representation is called "the distributional representation". We obtain the distributional representation of the generalized hypergeometric functions which leads to some new integral formulas about generalized hypergeometric functions as well as for Gauss and confluent hypergeometric functions. An application of the distributional representation gives a formula which can be considered as a generalization of Ramanujan's master theorem. New proofs of Euler's reflection formula and the functional equation for the Riemann zeta function are presented based on the distributional representation.The generalized gamma, the extended beta, the extended Gauss hypergeometric and the extended confluent hypergeometric functions have been defined and proved to be useful in several applications. We *apply Parseval's identity for the Mellin transform to these functions. Several integrals of products involving these extended functions have been obtained. Some applications of Parseval's formula for the incomplete Mellin transform are discussed. We obtain a generalization of the extended Fermi-Dirac and extended Bose-Einstein functions by inserting a regularizing factor in the integral representations. These generalizations also provide some results for the original Fermi-Dirac and Bose-Einstein functions as well as for other zeta family functions.*

الخلاصة

الاسم: **محمد حسين محمد آل ليل**

عنوان الدراسة: **التمثيل التوزيعي وتحويل فورير مع بعض التطبيقات**

التخصص: **الرياضيات**

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لقد قمنا باستخدام تحويلات فورير للدوال فوق الهندسية المعممة وذلك لإيجاد قيم لتكاملات حاصل ضرب دالتين فوق هندسيتين. كُتب الناتج بدلالة دالة كام دا فيري (Kampé de Fériet) المعممة فوق الهندسية ومتعددة المتغيرات. العديد من المتطابقات الخاصة بتكاملات دالة جاوس فوق الهندسية تم اشتقاقها كحالات خاصة. أثبتنا أنه يمكن كتابة أي داله لها تحويل ميلن (Mellin transform) كمجموع لدالة ديراك دلتا، حيث يسمى هذا التمثيل بالتمثيل التوزيعي للدالة. قمنا بإيجاد التمثيل التوزيعي للدوال فوق الهندسية المعممة والذي قادنا لإيجاد بعض الصيغ الجديدة لتكاملات خاصة بهذه الدوال المعممة علاوة على صيغ جديدة لدالة جاوس وكونفلونت فوق الهندسية. أحد تطبيقات التمثيل التوزيعي أعطانا صيغة معممة للنظرية الاساسية لرمانوجن (Ramanujan's master theorem). كما أمكننا استخدام التمثيل التوزيعي لإيجاد إثبات جديد لصيغة أويلر العاكسة لدالة جاما (Euler's reflection formula) وإثبات جديد للصيغة العاكسة لدالة زيتا لريمان (Riemann's functional equation). تعميم الدوال الخاصة مثل دوال جاما وبيتا و جاوس فوق الهندسية اثبتت فائدتها في كثير من التطبيقات. قمنا باستخدام متطابقة بارسفال (Parseval's identity) لتحويل ميلن على هذه الدوال الخاصة المعممة ومنها استطعنا الحصول على بعض صيغ تكاملات حاصل ضرب دالتين منها. كذلك تم نقاش بعض تطبيقات متطابقة بارسفال لتحويل ميلن الناقص. وأخيراً قمنا بتعريف امتداد جديد لتعميمي دالة ديراك فيرمي ودالة بوس أينشتاين وذلك بإضافة عامل *استقرار* للصيغة التكاملية. هذه الامتدادات الجديدة اعطتنا بعض النتائج لدالة ديراك فيرمي ودالة بوس أينشتاين الأصليتين بالإضافة لبعض النتائج لباقي عائلة دالة زيتا لريمان.

CHAPTER 1

INTRODUCTION AND PRELIMINARIES

1.1 Introduction

Special functions have been used for centuries. The Greeks used trigonometric ratios for measurement and for astronomy. Special functions appear as solutions of several physical problems. Because of their remarkable properties, the subject of special functions has been continuously developed. Some of the greatest mathematicians, including Euler, Legendre, Laplace, Gauss, Kummer, Riemann, and Ramanujan have contributed to the subject of special functions.

Many special functions arise as solutions of second order differential equations of mathematical physics and engineering. These include the Legendre polynomials and functions, the Laguerre, the Hermite, and the Bessel functions. All of these functions are special cases of the hypergeometric, or confluent hypergeometric functions [36] which are solutions of the hypergeometric differential equation

$$z(1-z)\frac{d^2u}{dz^2} + [c-(a+b+1)z]\frac{du}{dz} - abu = 0.$$

Another collection of special functions are those that arose in the theory of numbers: the gamma function and the family of zeta functions. In the 1720s, Euler defined the gamma function as a generalization of the factorial function by extending its domain from integers to the real and thence to the complex numbers. Many special functions can be defined in terms of the gamma function. In 1737, Euler related the sum of the reciprocals of the positive integers to the product of primes. The zeta function, though originally introduced by Euler, was used by Riemann to solve a problem in the theory of prime numbers [16]. Riemann denoted the zeta function by $\zeta(s)$. The Riemann zeta function was originally defined for $\Re(s) > 1$ (real part of s greater than 1) and can be analytically continued to the whole complex plane with a simple pole at $s = 1$. It has simple zeros at the negative even integers, which are called the trivial zeros. All other zeros of the Riemann zeta function, which are infinitely many are called the nontrivial zeros. Studying the properties of the zeta function, Riemann conjectured that the nontrivial zeros of $\zeta(s), s = \sigma + i\tau$, lie on the critical line $\sigma = \frac{1}{2}$ in the complex plane. Though it is proved that they are restricted to the strip, $0 < \sigma < 1$, there is still no proof of the *Riemann hypothesis*. The Riemann zeta function has played an important role in Analytic Number Theory. It has different generalizations in the literature, for example, the polylogarithm, the Hurwitz zeta, the Hurwitz-Lerch zeta functions [18]. Riemann's zeta function and its extensions have many applications in different areas of mathematics and physics.

Integral transformations have been used for solving many problems in applied mathematics, mathematical physics, and engineering science. The origin of the integral transforms can be traced back to the work of Laplace (1749–

1827) and Fourier (1768–1830). The Laplace transform has been used in finding solution of linear differential equations and integral equations. Fourier's work provided the modern mathematical theory of Fourier series, Fourier integrals and heat conduction. Fourier proved that an arbitrary function defined on a finite interval can be expanded in terms of trigonometric series which is known as the Fourier series. In an attempt to extend his ideas to functions defined on an infinite interval, Fourier discovered an integral transform and its inversion formula which are now well known as the Fourier transform (FT) and the inverse Fourier transform. Many linear boundary value and initial value problems in applied mathematics, mathematical physics, and engineering science can be solved by using the Fourier transform.

There are many other integral transformations including the Mellin transform, the Hankel transform and the Hilbert transform which are widely used to solve initial and boundary value problems involving ordinary and partial differential equations and other problems in mathematics, science and engineering. The Mellin transform can be used in finding sums of infinite series.

Distributions (or generalized functions) have many applications in physics and engineering. They are defined as continuous linear functionals over a space of test functions. The most commonly encountered generalized function is the Dirac delta function, $\delta(x)$. A nice treatment of distributions from a physicist's point of view is given by Vladimirov in [47]. The Fourier transform representation of a function leads to its series representation in terms of Dirac delta functions of complex argument. This series representation is called "distributional representation" here.

An important aspect of the analysis of special functions is to find their properties. There is more than one representation for special functions, for example, the series representation, the asymptotic representation and the integral representation. The integral of a product of special functions can produce a new special function, which may prove more useful than the original functions. The tables of integrals contain a large number of integrals involving the gamma function, Bessel functions, Legendre polynomials, hypergeometric and related functions. Only a few integrals of the Riemann zeta and related functions have been found in these tables. In this thesis, we obtain some integrals involving the Riemann zeta functions.

Definite integrals of products of two generalized hypergeometric functions have numerous applications in pure and applied mathematics (see, for example, [19]). Not all such integrals have been available in the mathematical literature. In this thesis, we use the FT representations of the generalized hypergeometric functions to evaluate integrals involving generalized hypergeometric functions. A number of new integral identities for confluent and Gauss hypergeometric functions are presented.

The FT had long been used for representation of some well known "special functions", but the distributional representations were only introduced relatively recently [6]. At first sight it might seem that they could not be used for analytic functions as distributions provide real measures for functions. This limitation was removed in a somewhat ad hoc manner in [6] by extending the range of the functional to the space of complex numbers, $\mathbb{C}$, and was later provided a rigorous basis in [40] and [42]. The FT and distributional representations led to new results for integrals involving the Riemann zeta function and other members of the zeta family of functions [42]. Due to greater familiarity (and hence acceptance) of readers with the FT representation it had been used far more extensively for final presentation than the distributional representation. However, many of the results could equally well or more easily be obtained by the distributional representation. There is an elegance to the latter, which makes it worth using in preference to the FT representation. In this thesis, we present a distributional representation of the generalized hypergeometric functions, which is then used to evaluate integrals involving generalized hypergeometric functions. A number of new integral identities for confluent and Gauss hypergeometric functions are found.

Fermi-Dirac (FD) and Bose Einstein (BE) functions arise in many physics problems. They arose in the distribution functions for Quantum Statistics [24]. The FD functions were first encountered in the 1920s, when Pauli and Sommerfeld used them to describe the degenerate electron gas of a metal. The BE functions, whose mathematical theory is also well developed, are widely used in connection with the theory of the Bose–Einstein gas above its transition temperature and in the theory of ferromagnetism. The FD and BE functions have close connection with the zeta function. Due to their physical significance, their mathematical properties, asymptotic expansions, relations to other functions, generalizations have been extensively studied.

Generalization of special functions may prove more useful than the original special functions themselves [35]. For an acceptable generalization it is requisite that the results for the generalization should be no less elegant than those for the original function. In certain situations, while the original functions do not solve the problem the generalization does. Some generalizations of the gamma, zeta, error, beta and hypergeometric functions with applications are discussed in [10, 11, 35] and references therein. In this thesis we apply the Parseval formula for Mellin transform to generalized gamma, extended beta, extended Gauss hypergeometric and extended confluent hypergeometric functions.

The FD and BE functions are extended in [39]. Here, we generalize these extended functions by introducing an extra parameter. We give some applications and identities for these functions.

This thesis is organized as follows: The rest of Chapter **1** is devoted to the definitions and fundamental results. In Chapter **2**, we use the FT representation of the generalized hypergeometric function to evaluate integrals of products of two generalized hypergeometric functions. In Chapter **3**, we obtain the distributional representation of the generalized hypergeometric functions, we use this representation to evaluate integrals involving generalized hypergeometric, confluent and Gauss hypergeometric functions. After that, we obtain the distributional representation for a Mellin transformable function with some applications. In Chapter **4,** we apply the Parseval formula for Mellin transform to generalized gamma, extended beta, extended Gauss hypergeometric, and extended confluent hypergeometric functions. Some applications of Parseval's formula for incomplete Mellin transform are presented.

In Chapter **5**, the generalized extended Fermi-Dirac and Bose-Einstein functions are introduced with some applications.

The last Chapter **6** gives the Conclusion and suggestions for further studies related to the research discussed in this thesis.

1.2 Gamma and generalized gamma functions

The problem of extending the definition of the factorial functions, $x!$ was the starting point that led to the gamma function. Euler (1707-1783) defined the gamma *function* by

$$\Gamma(s) := \int_0^\infty t^{s-1} e^{-t} dt \qquad (\Re(s) > 0), \tag{1.1}$$

where $\Re(s)$ stands for the real part of s. When s is a positive integer, $\Gamma(s)$ equals $(s-1)!$ The integral in (1.1) can be written as a sum of two integrals

$$\Gamma(s) = \int_0^1 t^{s-1}e^{-t}dt + \int_1^\infty t^{s-1}e^{-t}dt. \tag{1.2}$$

The first integral defines a function, which is analytic in the half plane $\Re(s) > 0$, while the second integral defines an entire function. The gamma function can be analytically continued to the whole complex plane excluding the points $s = 0, -1, -2, \ldots$, where it has simple poles. To explicitly represent the poles and the analytic continuation of the gamma function, we expand e^{-t} into Taylor series in the first integral in (1.2) and integrate term by term we get

$$\Gamma(s) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!(s+n)} + \int_1^\infty t^{s-1}e^{-t}dt \quad (s \neq 0, -1, -2, ...). \tag{1.3}$$

The second function on the right-hand side of (1.3) is an entire function, and the first shows that the poles are as claimed, with $1, -1, 1/2, ...(-1)^n/n!, ...$ being the residue at $s = 0, -1, -2, \ldots$ Many special functions can be defined in terms of the gamma function, which makes it one of the simplest and most important special function. The gamma function has many properties. We give the ones that we will use in our study. The gamma function satisfies the recurrence formula

$$\Gamma(s+1) = s\,\Gamma(s), \tag{1.4}$$

which can be shown by integrating (1.1) by parts. With $\Gamma(1) = 1$, it gives the following relation between gamma function and factorial

$$\Gamma(n+1) = n!, \qquad (n = 0, 1, 2, \ldots).$$

It is possible to show that [11]

$$\Gamma(s)\Gamma(1-s) = \frac{\pi}{\sin(\pi s)} \qquad (s \neq 0, \pm 1, \pm 2, \ldots), \tag{1.5}$$

which is called the reflection formula. As a consequence of (1.5), $\Gamma(1/2) = \sqrt{\pi}$. With the recurrence formula, one can further determine the values $\Gamma(3/2)$, $\Gamma(5/2)$, ... While the gamma function is not analytic for the non-positive integers, its reciprocal, $1/\Gamma(s)$, is analytic for all s. The duplication formula is [11]

$$\Gamma(s)\Gamma\left(s+\frac{1}{2}\right) = 2^{1-2s}\sqrt{\pi}\,\Gamma(2s) \qquad (\Re(s) > 0). \tag{1.6}$$

Note that if s is an integer n, then the duplication formula is written as

$$\Gamma\left(n+\frac{1}{2}\right) = \frac{\sqrt{\pi}(2n)!}{2^{2n}\,n!} \quad (n = 0, 1, 2, 3, \ldots). \tag{1.7}$$

Finally, the familiar Stirling formula [11]

$$\Gamma(s) = \sqrt{2\pi} s^{s-1/2} e^{-s}\left[1 + O(\frac{1}{s})\right] \qquad (\text{as } |s| \to \infty \text{ with } |\arg s| < \pi), \tag{1.8}$$

is an asymptotic approximation to $\Gamma(s)$ for large complex arguments. The Stirling formula uses the principal branch of log for $s^{s-1/2}$.

The *digamma* or the *psi* function, denoted by $\psi(s)$, is the logarithmic derivative of the gamma function. It is defined in [11]

$$\psi(s) := \frac{d}{ds}\{\ln\Gamma(s)\} = \frac{\Gamma'(s)}{\Gamma(s)}. \tag{1.9}$$

The digamma function satisfies the following properties:

$$\psi(s+1) = \psi(s) + \frac{1}{s}, \tag{1.10}$$

$$\psi(s) - \psi(1-s) = -\frac{\pi}{\tan \pi s} \quad (s \neq 0, \pm 1, \pm 2, \ldots), \tag{1.11}$$

$$\psi(n+1) = -\gamma + (1 + \frac{1}{2} + \frac{1}{3} + ... + \frac{1}{n}) \quad (n = 0, 1, 2, ...), \tag{1.12}$$

where

$$\gamma := \lim_{n\to\infty} \sum_{k=1}^{n} \left(\frac{1}{k} - \ln(n) \right) = 0.57721566..., \tag{1.13}$$

is Euler's constant.

The *beta* function is defined by

$$B(\alpha, \beta) := \int_0^1 t^{\alpha-1}(1-t)^{\beta-1} dt \qquad (\Re(\alpha), \Re(\beta) > 0). \tag{1.14}$$

By using the substitution $t = 1 - s$, we easily obtain the symmetry

$$B(\alpha, \beta) = B(\beta, \alpha).$$

The connection between the beta function and the gamma function is given by

$$B(\alpha, \beta) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha + \beta)} \quad (\Re(\alpha), \Re(\beta) > 0). \tag{1.15}$$

We note that the substitution $t = \frac{x}{1-x}$ in (1.14) gives another representation of the beta function that is

$$B(\alpha, \beta) = \int_0^\infty \frac{t^{\alpha-1}}{(1+t)^{\alpha+\beta}}\, dt \qquad (\Re(\alpha), \Re(\beta) > 0). \tag{1.16}$$

The gamma function can be broken into the two incomplete gamma functions [11, p. 9] by breaking the range of integration for Eq. (1.1) at some finite positive number x

$$\gamma(\alpha, x) := \int_0^x t^{\alpha-1}\, e^{-t}\, dt \qquad (\Re(\alpha) > 0, \quad |arg(\alpha)| < \pi), \tag{1.17}$$

$$\Gamma(\alpha, x) := \int_x^\infty t^{\alpha-1}\, e^{-t}\, dt \qquad (|arg(\alpha)| < \pi). \tag{1.18}$$

So that

$$\gamma(\alpha, x) + \Gamma(\alpha, x) = \Gamma(\alpha).$$

It is possible to extend the classical gamma function in infinitely many ways. Some of these extensions could be useful in certain types of problems. However, it is desirable to find an extension of the gamma function that meets the requirement that the previous results for the function are naturally and simply extended. It is also required that the results for the extension should be no less elegant than those for the original function [35]. Chaudhry and Zubair gave a generalization of the Euler gamma function. This generalization is given in [11, p. 9].

$$\Gamma_b(s) = \int_0^\infty t^{s-1}\, e^{-t-b/t}\, dt \qquad (\Re(b) \geq 0, \Re(s) > 1 \text{ if } b = 0). \tag{1.19}$$

The factor $e^{-b/t}$ in the integral (1.19) plays the role of a regularizer. For $\Re(b) > 0$, $\Gamma_b(s)$ is defined in the whole complex plane and for $b = 0$, the function $\Gamma_b(s)$ is defined in the half plane $\Re(s) > 0$. The integral in (1.19) can be simplified in terms of the Macdonald function to give [11, p. 9, Eq. (1.67)]

$$\Gamma_b(s) = 2b^{\frac{s}{2}} K_s(2\sqrt{b}) \quad (\Re(b) > 0, \, |arg(\sqrt{b})| < \pi). \tag{1.20}$$

The Macdonald function appears as a special case of the Bessel functions for imaginary argument.

Several properties of the generalized gamma function can be proved by using the representation (1.20) together with the properties of the Macdonald function. The difference formula for the generalized gamma is given in [11, p. 10, Eq. (1.73)]

$$\Gamma_b(s+1) = s\Gamma_b(s) + b\,\Gamma_b(s-1), \tag{1.21}$$

and the reflection formula is given in [11, p. 13, Eq. (1.88)]

$$b^s\Gamma_b(-s) = \Gamma_b(s) \qquad (\Re(b) > 0). \tag{1.22}$$

The generalized gamma function $\Gamma_b(s)$ has been found to be a simple and natural generalization of the Euler gamma function. This generalization leads to a generalization of the Psi (digamma) function. Analogous to the definition of the digamma function, the generalized digamma function is defined as the logarithmic derivative of the generalized gamma function [11, p. 23, Eq. (1.169)]

$$\psi_b(s) := \frac{d}{ds}\left\{\ln\left(\Gamma_b(s)\right)\right\} = \frac{1}{\Gamma_b(s)}\,\frac{d}{ds}\left\{\Gamma_b(s)\right\}. \tag{1.23}$$

From the integral representation (1.19) of the generalized gamma function we have [11, p. 23, Eq. (1.170)]

$$\psi_b(s) = \frac{1}{\Gamma_b(s)}\int_0^\infty t^{s-1}(\ln t)e^{-t-bt^{-1}}\,dt \qquad (\Re(b) \geq 0, \Re(s) > 1 \text{ if } b = 0). \tag{1.24}$$

The reflection formula of the generalized digamma function is given in [11, p. 23, Eq. (1.171)]

$$\psi_b(-\alpha) = \ln b - \psi_b(\alpha) \qquad (\Re(b) > 0). \tag{1.25}$$

For $b = 0$, it reduces to the digamma function.

The extended beta function is defined in [11, p. 221]

$$B(s,w;p) := \int_0^1 t^{s-1}(1-t)^{w-1}\exp\left[\frac{-p}{t(1-t)}\right]dt = B(w,s;p)$$
$$(\Re(p)\geq 0, \Re(s), \Re(w) > 0 \text{ if } p=0). \tag{1.26}$$

For extensive applications of the generalized gamma function to a wide variety of problems, we refer to [11].

1.3 The hypergeometric functions

Many of the elementary functions of mathematics are either hypergeometric or ratios of hypergeometric functions [31].

The generalized hypergeometric function is defined by [31, p. 155]

$${}_pF_q\left[a_1,\cdots,a_p;b_1,\cdots,b_q;z\right] := \sum_{n=0}^{\infty}\frac{(z)^n(a_1)_n\cdots(a_p)_n}{n!(b_1)_n\cdots(b_q)_n}, \tag{1.27}$$

where $(\lambda)_n := \frac{\Gamma(\lambda+n)}{\Gamma(\lambda)}$ is the Pochhammer symbol. The series in (1.27) converges absolutely for all z if $p\leq q$ and for $|z|<1$ if $p=q+1$, and diverges for all $z\neq 0$ if $p>q+1$. Many functions that arise in physics and engineering are special cases of generalized hypergeometric functions.

The generalized hypergeometric functions have the following integral representations [31, p. 161]:

$${}_pF_q[a_1,\cdots,a_p;b_1,\cdots,b_q;z] = \frac{1}{\Gamma(a_p)}\int_0^{\infty} t^{a_p-1}e^{-t}\ {}_{p-1}F_q[(a_{p-1});(b_q);zt]dt$$
$$(\Re(a_p)>0, p\leq q+1, |z|<1 \text{ if } p=q+1) \tag{1.28}$$

Some of the generalized hypergeometric functions have special names. For example, ${}_1F_1[a;b;z]$ also denoted by $\Phi[a;b;z]$ is called a confluent hypergeometric function (CHF) and satisfies the following, Kummer's, differential equation [31, p. 287]:

$$z\frac{d^2u}{dz^2} + [b - z]\frac{du}{dz} - au = 0.$$

The function ${}_2F_1[a, b; c; z]$ is called the hypergeometric function or Gauss's hypergeometric function (GHF), it is sometimes denoted simply by $F[a, b; c; z]$. It satisfies the following hypergeometric differential equation [31, p. 260]:

$$z(1 - z)\frac{d^2u}{dz^2} + [c - (a + b + 1)z]\frac{du}{dz} - abu = 0.$$

Choosing the parameters a, b, c in ${}_1F_1[a; b; z]$ and ${}_2F_1[a, b; c; z]$, appropriately, most of the special functions of mathematical physics such as the Bessel, Laguerre and Legendre functions can be obtained.

Many of the elementary functions have representations as hypergeometric series. For example

$$e^x = {}_0F_0[-; -; x]; \tag{1.29}$$

$$\sin x = x\,{}_0F_1[-; 3/2; \frac{-x^2}{4}]; \tag{1.30}$$

$$\cos x = {}_0F_1[-; 1/2; \frac{-x^2}{4}]; \tag{1.31}$$

$$\sin^{-1} x = x\,{}_2F_1[1/2, 1/2; 3/2; x^2]; \tag{1.32}$$

$$\tan^{-1} x = x\,{}_2F_1[1/2, 1; 3/2; -x^2]; \tag{1.33}$$

$$\log(1 + x) = x\,{}_2F_1[1/2, 1; 3/2; -x^2]; \tag{1.34}$$

$$(1 - x)^{-a} = {}_1F_0[a, ; -; x]. \tag{1.35}$$

In [10] Chaudhry et al. used $B(s, w; p)$ to extend the hypergeometric functions and confluent hypergeometric functions as

$$F_p[a,b;c;z] := \sum_{n=0}^{\infty} \frac{(a)_n B(b+n, c-b; p)}{B(b, c-b)} \frac{z^n}{n!}$$
$$(\Re(p) \geq 0, |z| < 1, \Re(c) > \Re(b) > 0), \tag{1.36}$$

$$\phi_p[b;c;z] := \sum_{n=0}^{\infty} \frac{B(b+n, c-b; p)}{B(b, c-b)} \frac{z^n}{n!}$$
$$(\Re(p) \geq 0, \Re(c), \Re(b) > 0), \tag{1.37}$$

respectively. They called these functions extended Gauss hypergeometric function (EGHF) and extended confluent hypergeometric function (ECHF), respectively. They provided the integral representations of EGHF as

$$F_p[a,b;c;z] = \frac{1}{B(b, c-b)} \int_0^1 \frac{t^{b-1}(1-t)^{c-b-1}}{(1-zt)^a} \exp\left[-\frac{p}{t(1-t)}\right] dt$$
$$(\Re(c) > \Re(b) > 0, \Re(p) \geq 0, |\arg(1-z)| < \pi \text{ if } p = 0), \tag{1.38}$$

and

$$F_p[a,b;c;z] = \frac{e^{-2p}}{B(b, c-b)} \int_0^\infty \frac{t^{b-1}(1-t)^{a-c}}{[1+t(1-z)]^a} \exp\left[-p\left(t + \frac{1}{t}\right)\right] dt$$
$$(\Re(c) > \Re(b) > 0, \Re(p) \geq 0, |\arg(1-z)| < \pi \text{ if } p = 0). \tag{1.39}$$

The integral representation of ECHF is given by

$$\phi_p[b;c;z] = \frac{1}{B(b, c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1} \exp\left[zt - \frac{p}{t(1-t)}\right] dt$$
$$(\Re(p) \geq 0, \Re(c) > \Re(b) > 0 \text{ if } p = 0). \tag{1.40}$$

By substituting $y = t/(1-t)$, one obtain

$$\phi_p[b;c;z] = \frac{\exp(-2p)}{B(b,c-b)} \int_0^\infty y^{b-1}(1+y)^{-c} \exp\left[z\left(\frac{y}{y+1}\right) - p\left(y + \frac{1}{y}\right)\right] dy$$

$$(\Re(p) \geq 0, \Re(c), \Re(b) > 0 \text{ if } p = 0). \tag{1.41}$$

They also discussed the differentiation properties and Mellin transforms and obtained transformation formulas, recurrence relations, summation and asymptotic formulas for these functions. Note that

$$F_0[a,b;c;z] = F[a,b;c;z] = {}_2F_1[a,b;c;z]$$

and

$$\phi_p[b;c;z] = \phi[b;c;z] = {}_1F_1[b;c;z].$$

The generalized Kampé de Fériet hypergeometric function of two variables is defined by [38, p. 63]

$$F^{p:q;k}_{l:r;s}\left[\begin{array}{c} (a_p):(b_q);(c_k); \\ (\alpha_l):(\beta_r);(\gamma_s); \end{array} x, y\right]$$

$$= \sum_{m,n=0}^{\infty} \frac{\prod_{j=1}^{p}(a_j)_{m+n} \prod_{j=1}^{q}(b_j)_m \prod_{j=1}^{k}(c_j)_n}{\prod_{j=1}^{l}(\alpha_j)_{m+n} \prod_{j=1}^{r}(\beta_j)_m \prod_{j=1}^{s}(\gamma_j)_n} \frac{x^m}{m!} \frac{y^n}{n!}, \tag{1.42}$$

where (λ_p) denotes the array of p parameters $\lambda_1, \lambda_2, \cdots \lambda_p$. The series in (1.42) converges absolutely for

1. $p+q < l+r+1$, $p+k < l+s+1$, $|x| < \infty$, $|y| < \infty$, or for
2. $p+q = l+r+1$, $p+k = l+s+1$, and

$$\begin{cases} |x|^{1/(p-l)} + |y|^{1/(p-l)} < 1, \text{ if } p > l, \\ \max\{|x|, |y|\} < 1, \text{ if } p \leq l. \end{cases}$$

1.4 The family of zeta functions

The Riemann zeta function is one of the most important functions in analytic number theory. It was originally defined for real arguments as

$$\zeta(x) := \sum_{n=1}^{\infty} \frac{1}{n^x} \qquad (x > 1).$$

Euler, in 1737, found another formula for the zeta function, namely

$$\zeta(x) = \prod_{p \text{ prime}} \left(1 - \frac{1}{p^x}\right)^{-1} \qquad (x > 1).$$

In 1859, Bernhard Riemann extended the domain of definition from real x to complex $s = \sigma + i\tau$ as

$$\zeta(s) := \sum_{n=1}^{\infty} \frac{1}{n^s} \qquad (\Re(s) > 1), \tag{1.43}$$

and

$$\zeta(s) := \prod_{p \text{ prim}} \left(1 - \frac{1}{p^s}\right)^{-1} \qquad (\Re(s) > 1). \tag{1.44}$$

Riemann showed that it can be analytically continued in the whole complex plane except for $s = 1$, where it has a simple pole with residue 1. The function plays a very significant role in the theory of the distribution of primes. One of the most important properties of the zeta function is the functional equation [46, p. 13]

$$\zeta(s) = 2^s \pi^{s-1} \sin(\frac{\pi s}{2}) \Gamma(1-s) \zeta(1-s) \tag{1.45}$$

that can also be written as

$$\pi^{-(s/2)}\Gamma(\frac{s}{2})\zeta(s) = \pi^{-(\frac{1-s}{2})}\Gamma(\frac{1-s}{2})\zeta(1-s). \tag{1.46}$$

It relates $\zeta(s)$ with $\zeta(1-s)$ and gives an extension of the Riemann zeta function on the entire complex plane $\mathbb{C}$ except $s = 1$. Some other methods for extension of the zeta function are discussed in [46].

The Riemann zeta function has the integral representation [46, p. 18]

$$\zeta(s) = \frac{1}{\Gamma(s)} \int_0^\infty \frac{t^{s-1}}{e^t - 1} dt \qquad (\Re(s) > 1), \tag{1.47}$$

which can be written as

$$\Gamma(s)\zeta(s) = \int_0^1 \left(\frac{1}{e^t - 1} - \frac{1}{t} \right) t^{s-1} dt + \frac{1}{s-1} + \int_1^\infty \frac{t^{s-1}}{e^t - 1} dt. \tag{1.48}$$

The formula (1.48) holds for $\Re(s) > 0$. For $0 < \Re(s) < 1$, one can use

$$\frac{1}{s-1} = -\int_1^\infty t^{s-2} dt = -\int_1^\infty \frac{t^{s-1}}{t} dt. \tag{1.49}$$

These two equations lead to another representation of the Riemann zeta function given by [46, p. 23]

$$\zeta(s) = \frac{1}{\Gamma(s)} \int_0^\infty \left(\frac{1}{e^t - 1} - \frac{1}{t} \right) t^{s-1} dt \qquad (0 < \Re(s) < 1). \tag{1.50}$$

By multiplying (1.43) with the factor $1 - 2^{1-s}$, one can get

$$\zeta(s) := \frac{1}{1 - 2^{1-s}} \sum_{n=1}^\infty \frac{(-1)^{n-1}}{n^s} \qquad (\Re(s) > 0), \tag{1.51}$$

which has the following integral representation

$$\zeta(s) = \frac{1}{C(s)} \int_0^\infty \frac{t^{s-1}}{e^t + 1} dt \qquad (\Re(s) > 0), \tag{1.52}$$

where

$$C(s) := \Gamma(s)(1 - 2^{1-s}). \tag{1.53}$$

Here the pole of the zeta function at $s = 1$ is canceled by the zero of $1 - 2^{1-s}$.

For $\Re(s) > 1$, the function $\zeta(s)$ does not vanish [46]. Using the functional equation (1.45) and the fact that $\Gamma(s) \neq 0$ for $\Re(s) > 0$, we see that $\zeta(s)$ vanishes in the half-plane $\Re(s) < 0$ only at the points where the sine function is zero, namely from (1.45) we obtain

$$\zeta(-2k) = 2^{-2k}\pi^{-2k-1}\sin(-\frac{2k\pi}{2})\Gamma(1 + 2k)\zeta(1 + 2k) = 0 \ (k = 1, 2, \ldots). \tag{1.54}$$

These are simple zeros, called the *trivial zeros*. All the other zeros, called the non-*trivial zeros,* of the Riemann zeta function are symmetric about the critical line $\Re(s) = 1/2$ in the critical strip $0 < \Re(s) < 1$. The multiplicity of these non-trivial zeros (in general) is not known. Riemann conjectured that all the non trivial zeros of the Riemann zeta function lie on the critical line $\Re(s) = 1/2$. This conjecture is called the Riemann hypothesis. It is the most famous open problem in mathematics. It was shown by Hardy [16] that an infinite number of zeros lie on the critical line $\Re(s) = 1/2$. There is strong numerical evidence that the Riemann hypothesis is true but there is still no proof [13, 14, 30].

There have been several generalizations of the Riemann zeta function, like the Hurwitz zeta function defined in [37, Chapter 2]

$$\zeta(s, a) := \sum_{n=0}^{\infty} \frac{1}{(n + a)^s} \qquad (\Re(s) > 1, a \neq 0, -1, -2, \ldots). \tag{1.55}$$

It has the following integral representation

$$\zeta(s,a) = \frac{1}{\Gamma(s)} \int_0^\infty \frac{e^{-(a-1)t} t^{s-1}}{e^t - 1} dt \qquad (\Re(a) > 0, \Re(s) > 1), \tag{1.56}$$

and it is related to Riemann zeta function by

$$\zeta(s) = \zeta(s,1) = \frac{1}{2^{s-1}} \zeta(s, \frac{1}{2}) = 1 + \zeta(s,2). \tag{1.57}$$

Hurwitz zeta function has another series representation [37, p. 144]

$$\sum_{n=0}^{\infty} \frac{(s)_n}{n!} \zeta(s+n,a) t^n = \zeta(s, a-t) \qquad (|t| < |a|),$$

which holds true by the principle of analytic continuation, for all values of $s \neq 1$.

Another important generalization of the Riemann zeta function is the polylogarithm function

$$Li_s(x) := \phi(x,s) := \sum_{n=1}^{\infty} \frac{x^n}{(n)^s}, \tag{1.58}$$

which is analytic in the region $|x| \leq 1 - \rho$, for all s and each $0 < \rho < 1$. At the point $x = 1$, it converges for $\Re(s) > 1$ and

$$\phi(1,s) = \zeta(s).$$

The integral representation for the polylogarithm function is given by

$$Li_s(x) = \frac{x}{\Gamma(s)} \int_0^\infty \frac{t^{s-1}}{e^t - x} dt. \tag{1.59}$$

If x lies anywhere except on the segment of the real axis from 1 to ∞, where a cut is imposed then (1.59) defines an analytic function of x provided $\Re(s) > 0$.

For $x = 1$ and $\Re(s) > 1$, one obtains the special case

$$\phi(1, s) = \zeta(s) = \frac{1}{\Gamma(s)} \int_0^\infty \frac{t^{s-1}}{e^t - 1} dt.$$

The polylgarithm function is further generalized to the *Hurwitz-Lerch zeta function* [18, p. 27] by

$$\Phi(z, s, a) := \sum_{n=0}^{\infty} \frac{z^n}{(n + a)^s}$$
$$(|z| < 1, a \neq 0, -1, -2, \dots, \Re(s) > 1, \text{ if } |z| = 1). \tag{1.60}$$

This function has the integral representation [18, p. 27]

$$\Phi(z, s, a) = \frac{1}{\Gamma(s)} \int_0^\infty \frac{e^{-(a-1)t} t^{s-1}}{e^t - z} dt$$
$$(\Re(a) > 0 \text{ and either } |z| \leq 1, z \neq 1, \Re(s) > 0 \text{ or } z = 1, \Re(s) > 1). \tag{1.61}$$

Many functions can be expressed in terms of the general Hurwitz-Lerch zeta function Φ. For example the polylogarithm function

$$Li_s(x) := x\Phi(x, s, 1),$$

the Hurwitz zeta function

$$\zeta(s, a) = \Phi(1, s, a),$$

and the Riemann zeta function

$$\zeta(s) = \Phi(1, s, 1),$$

are special cases of the Hurwitz-Lerch zeta function.

Also, the general Hurwitz-Lerch zeta function Φ is related to the Lerch zeta function

$$l_s(\xi) := \sum_{n=1}^{\infty} \frac{e^{2n\pi i\xi}}{(n)^s} = e^{2\pi i\xi}\Phi(e^{2\pi i\xi}, s, 1) \qquad (\xi \in \mathbb{R}, \Re(s) > 1), \tag{1.62}$$

and the Lipschitz-Lerch zeta function defined by [37, p. 122]

$$\phi(\xi, a, s) := \sum_{n=0}^{\infty} \frac{e^{2n\pi i\xi}}{(n+a)^s} = \Phi(e^{2\pi i\xi}, s, a)$$

$$(a \neq 0, -1, -2, \ldots, \Re(s) > 0 \text{ when } \xi \in \mathbb{R}\backslash\mathbb{Z} \text{ and } \Re(s) > 1 \text{ when } \xi \in \mathbb{Z}). \tag{1.63}$$

Using the elementary series identity for absolutely convergent series

$$\sum_{k=1}^{\infty} f(k) = \sum_{j=1}^{r} \left(\sum_{k=0}^{\infty} f(rk+j) \right) \qquad (r \in \mathbb{N}^*),$$

it was shown [37, eq. (3.8)] that

$$\Phi(z, s, a) = r^{-s} \sum_{j=1}^{r} \Phi(z^r, s, \frac{a+j-1}{r}) z^{j-1}. \tag{1.64}$$

For $z = 1$, it reduces to the identity

$$\zeta(s, a) = r^{-s} \sum_{j=1}^{r} \zeta(s, \frac{a+j-1}{r}). \tag{1.65}$$

For further properties of the Riemann zeta and related functions we refer to [1, 2, 3, 16, 25, 43, 44, 46].

Apart from these generalizations of the Riemann zeta function, the extended Riemann and Hurwitz zeta functions are defined in [11, Chapter 7] by introducing a regularizer $e^{-b/t}$ in integral representations (1.47), (1.52) and (1.56).

The extended Riemann zeta function is defined in [11, p. 298]

$$\zeta_b(s) := \frac{1}{\Gamma(s)} \int_0^\infty \frac{e^{-t-\frac{b}{t}}}{1-e^{-t}} t^{s-1} dt \qquad (\Re(b) \geq 0, \Re(s) > 1 \text{ if } b = 0). \tag{1.66}$$

Many properties of the Riemann zeta function can be obtained as a special cases of the properties of the extended Riemann zeta function. But it fails to extend these properties in the critical strip. For this purpose, the following extension was defined in [11, p. 305]

$$\zeta_b^*(s) := \frac{1}{C(s)} \int_0^\infty \frac{e^{-t-\frac{b}{t}}}{1+e^{-t}} t^{s-1} dt \qquad (\Re(b) \geq 0, \Re(s) > 0 \text{ if } b = 0). \tag{1.67}$$

where $C(s)$ is as defined by (1.53). These two extensions are related by

$$\zeta_b^*(s) = \frac{\zeta_b(s) - 2^{1-s}\zeta_{2b}(s)}{1-2^{1-s}}.$$

Also, it can be seen from (1.52) and (1.67) that

$$\zeta_0^*(s) = \zeta(s) \qquad (\Re(s) > 0).$$

Similar *extensions* for the Hurwitz zeta function are defined in [11, p. 308]

$$\zeta_b(s,\nu) := \frac{1}{\Gamma(s)} \int_0^\infty \frac{e^{-\nu t - b/t}}{1-e^{-t}} t^{s-1} dt \quad (0 < \nu \leq 1, \Re(b) \geq 0, \Re(s) > 1 \text{ if } b = 0), \tag{1.68}$$

$$\zeta_b^*(s,\nu) := \frac{1}{C(s)} \int_0^\infty \frac{e^{-\nu t - b/t}}{1+e^{-t}} t^{s-1} dt \quad (0 < \nu \leq 1, \Re(b) \geq 0, \Re(s) > 0 \text{ if } b = 0), \tag{1.69}$$

where $C(s)$ is as defined by (1.53).

1.5 Fermi-Dirac and Bose-Einstein functions

The Fermi-Dirac and Bose-Einstein functions arise in many physics problems. They arose in the distribution functions for quantum statistics. They come from the velocity distribution of a quantum gas. The Fermi-Dirac functions were first encountered in the 1920s, when Pauli and Sommerfeld used them to describe the degenerate electron gas of a metal. The Bose–Einstein functions, whose mathematical theory is also well developed, are widely used in connection with the theory of the Bose-Einstein gas above its transition temperature and in the theory of ferromagnetism.

The Fermi-Dirac (FD) function is defined in [18, p. 38]

$$F_{s-1}(x) := \frac{1}{\Gamma(s)} \int_0^\infty \frac{t^{s-1}}{e^{t-x}+1} dt \qquad (\Re(s) > 0), \tag{1.70}$$

and the Bose-Einstein (BE) function is defined in [18, p. 449]

$$B_{s-1}(x) := \frac{1}{\Gamma(s)} \int_0^\infty \frac{t^{s-1}}{e^{t-x}-1} dt \qquad (\Re(s) > 1). \tag{1.71}$$

The Fermi-Dirac and Bose-Einstein functions have been extended in such a way that they are closely related to the Riemann and other zeta functions [39]. These extensions have been studied and investigated by using Fourier transform and distributional representations in [40, 42]. The extended Fermi-Dirac (eFD) function is defined by

$$\Theta_\nu(s;x) := \frac{e^{-(\nu+1)x}}{\Gamma(s)} \int_0^\infty \frac{t^{s-1}e^{-\nu t}}{e^t + e^{-x}} dt \qquad (\Re(\nu) > -1, \Re(s) > 0, x \geq 0), \tag{1.72}$$

and the extended Bose-Einstein (eBE) function is defined by

$$\Psi_\nu(s;x) := \frac{e^{-(\nu+1)x}}{\Gamma(s)} \int_0^\infty \frac{t^{s-1}e^{-\nu t}}{e^t - e^{-x}} dt$$

$$(\Re(\nu) > -1, \Re(s) > 1 \text{ when } x = 0, \Re(s) > 0 \text{ when } x > 0). \tag{1.73}$$

For $\nu = 0$, these extended functions give the familiar FD and BE

$$\Theta_0(s;x) = F_{s-1}(-x) \qquad (\Re(s) > 0, x \geq 0), \tag{1.74}$$

$$\Psi_0(s;x) = B_{s-1}(-x) \qquad (\Re(s) > 1, x \geq 0). \tag{1.75}$$

These extended functions are related to the Riemann zeta function by

$$\zeta(s) = (1 - 2^{1-s})\Theta_0(s;0) \qquad (\Re(s) > 0),$$

and

$$\zeta(s) = \Psi_0(s;0) \qquad (\Re(s) > 1). \tag{1.76}$$

Also, the Hurwitz zeta function is a special case of eBE function

$$\zeta(s, \nu + 1) = \Psi_\nu(s;0) \qquad (\Re(s) > 1, \Re(\nu) > 1). \tag{1.77}$$

The Hurwitz-Lerch zeta function is related to the eFD and eBE functions by [39]

$$\Theta_\nu(s;x) = e^{-(\nu+1)}\Phi(-e^{-x}, s, \nu + 1) \tag{1.78}$$

and

$$\Psi_\nu(s;x) = e^{-(\nu+1)}\Phi(e^{-x}, s, \nu + 1). \tag{1.79}$$

1.6 The Mellin, Fourier and Weyl integral transforms

Integral transformations have been used in the study of various problems in applied mathematics, physics and engineering. For example the Fourier transform is used as a basic tool in such problems. Our interest here is to study Mellin, Fourier and Weyl transforms as they will be used in subsequent chapters.

Following the terminology of [29, p. 237] (see also [32, pp. 237-238]), classes of *good functions* $H(\kappa;\lambda)$ and $H(\infty;\lambda)$ are defined as follows:

A function $f \in C^{\infty}(0,\infty)$ is said to be a member of $H(\kappa;\lambda)$ if:

1. $f(t)$ is integrable on every finite subinterval $[0,T](0 < T < \infty)$ of $\mathbb{R}_0^+ := [0,\infty)$;

2. $f(t) = O(t^{-\lambda})(t \to 0^+)$;

3. $f(t) = O(t^{-\kappa})(t \to +\infty)$.

Furthermore, if the above relation $f(t) = O(t^{-\kappa})(t \to +\infty)$ is satisfied for every exponent $\kappa \in \mathbb{R}_0^+$, then the function $f(t)$ is said to be in the class $H(\infty;\lambda)$. It is to be noted that $H(\infty;\lambda) \subset H(\kappa;\lambda)$ $(\forall \kappa \in \mathbb{R}_0^+)$.

The Mellin transform of $f \in H(\kappa;\lambda)$ is defined by (see [17, Vol. I, Ch. 6] and [32, p. 79]).

$$M[f;s] = F_M(s) := \int_0^{\infty} t^{s-1} f(t)dt \qquad (\lambda < \Re(s) < \kappa). \tag{1.80}$$

This integral converges absolutely and defines an analytic function in the strip $\lambda < \Re(s) < \kappa$. This strip is known as the *strip of analyticity* of $M[f;s]$ and is denoted by S_f.

The inverse Mellin transform is given by

$$f(t) = \frac{1}{2\pi i}\int_{\alpha-i\infty}^{\alpha+\infty} F_M(s)t^{-s}ds \qquad (\lambda < \alpha < \kappa). \tag{1.81}$$

This formula is valid at all points $t \geq 0$ where $f(t)$ is continuous [45]. When the function $f(t)$ has a discontinuity, the inversion theorem takes the form (1.82). Suppose (1.80) converges absolutely on the line $\Re(s) = \alpha$ and let $f(y)$ be of bounded variation in a neighborhood of the point $y = t$. Then

$$\frac{1}{2}\left[f(t+0) + f(t-0)\right] = \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} F_M(s)t^{-s}ds. \tag{1.82}$$

Clearly, $f(t) = e^{-pt} \in H(\infty; 0)$ $(p > 0)$. The Mellin transform of $f(t)$ is equal, by definition, to

$$M[f; s] = \int_0^\infty t^{s-1}e^{-pt}dt.$$

Using the definition (1.1) of the gamma function, we obtain

$$M[f; s] = p^{-s}\Gamma(s) \qquad (p > 0, \Re(s) > 0).$$

As an example of a discontinuous function, let $f(t)$ be the simple step function given by

$$f(t) = \begin{cases} 1 & 0 < t < 1 \\ 0 & t > 1. \end{cases}$$

The Mellin transform of this function is

$$M[f(t), s) = \int_0^\infty t^{s-1}f(t)dt = \int_0^1 t^{s-1}dt = \frac{1}{s} \qquad (\Re(s) > 0). \tag{1.83}$$

Then from (1.82) we have

$$\frac{1}{2\pi i}\int_{\alpha-i\infty}^{\alpha+i\infty}\frac{t^{-s}}{s}ds = \begin{cases} 1 & 0<t<1 \\ \frac{1}{2} & t=1 \\ 0 & t>1, \end{cases} \tag{1.84}$$

provided $\alpha > 0$. An extension of (1.84) is given by

$$\frac{1}{2\pi i}\int_{\alpha-i\infty}^{\alpha+i\infty}\frac{t^{-s}}{s+q}ds = \begin{cases} t^q & 0<t<1 \\ \frac{1}{2} & t=1 \\ 0 & t>1, \end{cases} \tag{1.85}$$

provided that $\alpha > -q$.

The following properties hold for the Mellin transform:

- **Scaling:**

$$M[f(rt);s] = r^{-s}F_M(s) \qquad (r>0), \tag{1.86}$$

and

$$M[f(t^r);s] = r^{-1}F_M(r^{-1}s) \qquad (r>0). \tag{1.87}$$

- **Translation:**

$$M[t^a f(t);s] = F_M(s+a), \tag{1.88}$$

and

$$M[t^{-1}f(t^{-1});s] = F_M(1-s). \tag{1.89}$$

- **Differentiation:** If $f \in H(\infty;\lambda)$ is a k times differentiable function then

$$M[\frac{d^k}{dt^k}f(t);s] = (-1)^k(s-k)_k F_M(s-k) \quad (k\ = 1,2,...), \tag{1.90}$$

$$M[(x\frac{d}{dt})^k f(t);s] = (-s)^k F_M(s) \quad (k\ = 1,2,...). \tag{1.91}$$

- **Integration:**

$$M[\int_0^x f(t)dt; s] = -\frac{1}{s} F_M(s+1), \tag{1.92}$$

and

$$M[\int_x^\infty f(t)dt; s] = \frac{1}{s} F_M(s+1). \tag{1.93}$$

A fundamental result in Mellin transform theory is the *Parseval formula.* We assume that the Mellin transforms $M[f; 1-s] = F_M\,(1-s)$ and $M[g; s] = G_M(s)$ have a common strip of analyticity and we take the vertical line $\Re(s) = \alpha$ to lie in this common strip. Then, we have

$$\begin{aligned}
& \frac{1}{2\pi i} \int_{\alpha - i\infty}^{\alpha + i\infty} G_M\,(s)\, F_M\,(1-s)\, ds \\
= & \frac{1}{2\pi i} \int_{\alpha - i\infty}^{\alpha + i\infty} F_M\,(1-s) \left\{ \int_0^\infty x^{s-1} g(x) dx \right\} ds \\
= & \int_0^\infty g(x) \left\{ \frac{1}{2\pi i} \int_{\alpha - i\infty}^{\alpha + i\infty} x^{s-1} F_M\,(1-s)\, ds \right\} dx
\end{aligned}$$

upon interchanging the order of integration. If $F_M\,(1-\alpha-it) \in L(-\infty,\infty)$ and $x^{\alpha-1}g(x) \in L[0,\infty)$ then the interchange is justified by absolute convergence. We thus obtain Parseval's formula for the Mellin transform

$$\frac{1}{2\pi i} \int_{\sigma - i\infty}^{\sigma + i\infty} G_M\,(s)\, F_M\,(1-s)\, ds = \int_0^\infty f(x) g(x) dx. \tag{1.94}$$

Application of (1.88) in Parseval's formula shows that

$$\begin{aligned}
\int_0^\infty f(x) g(x) x^{p-1} dx &= \frac{1}{2\pi i} \int_{\sigma - i\infty}^{\sigma + i\infty} M[f(x); s] M[x^{p-1} g(x); 1-s] ds \\
&= \frac{1}{2\pi i} \int_{\sigma - i\infty}^{\sigma + i\infty} F_M\,(s)\, G_M\,(p-s)\, ds. \qquad (1.95)
\end{aligned}$$

Mellin's transformation is closely related to the Fourier integral transform.

The Fourier integral transform is defined by [48, Chapter 12]

$$\mathcal{F}[\varphi;\tau] := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{iy\tau}\varphi(y)dy \quad (\tau \in \mathbb{R}). \tag{1.96}$$

In this thesis, we will use Parseval's identity and duality property of the Fourier transform. Let f and g be Fourier transformable functions, the Parseval's identity [48, p. 232] of the Fourier Transform (FT) states that

$$\int_{-\infty}^{\infty} \mathcal{F}[f(y);\tau]\overline{\mathcal{F}[g(y);\tau]}d\tau = \int_{-\infty}^{\infty} f(y)\overline{g(y)}dy. \tag{1.97}$$

The *duality property* of the Fourier transform [13, p. 29] states that

$$\mathcal{F}[\mathcal{F}[\phi(y);\tau];\omega] = \phi(-\omega) \quad (\omega \in \mathbb{R}).$$

For further properties of the Fourier transform we refer to [13, 23, 48, 50].

Another integral transformation closely-related to the Mellin transform is known as the Weyl transform (or Weyl's fractional integral). The *Weyl transform* of order s of $\omega \in H(\kappa;0)$ is defined by (see [17, Vol II, Ch.13], [27] and [32, p. 236 et seq.]),

$$\begin{aligned}\Omega(s;x) := W_{x^+}^{s}[\omega(t)] &= \frac{1}{\Gamma(s)} \int_0^{\infty} t^{s-1}\omega(t+x)dt \\ &= \frac{1}{\Gamma(s)} \int_x^{\infty} (t-x)^{s-1}\omega(t)dt \qquad (0 < \Re(s) < \kappa, x \geq 0).\end{aligned} \tag{1.98}$$

The Weyl transform satisfies the semigroup property [32, p. 245]

$$W_{x^+}^{s+\beta}[\omega(t))] = W_{x^+}^{s}[\Omega(\beta;x)] = W_{x^+}^{\beta}[\Omega(s;x)],$$

or, equivalently,

$$\Omega(s+\beta;x) = W^{s}_{x^{+}}[\Omega(\beta;x)] = \frac{1}{\Gamma(s)}\int_{0}^{\infty} t^{s-1}\Omega(\beta;t+x)dt$$
$$= \frac{1}{\Gamma(s)}\int_{x}^{\infty}(t-x)^{s-1}\Omega(\beta;t)dt \quad (\min\{\Re(s),\Re(\beta)\}>0). \tag{1.99}$$

Also, if $\omega \in H(\kappa;0)$ and $\Omega(s;x)(x>0)$ is its Weyl transform, then

$$\Omega(s;x) = \sum_{n=0}^{\infty}\Omega(s-n;0)\frac{(-x)^{n}}{n!} \quad (0 \leq \Re(s) < \kappa, x > 0), \tag{1.100}$$

and

$$\omega(x) = \sum_{n=0}^{\infty}\Omega(-n;0)\frac{(-x)^{n}}{n!} \quad (0 \leq \Re(s) < \kappa, x > 0). \tag{1.101}$$

For further properties of Weyl's transform, we refer to [27].

1.7 Transformation of distributions

Distributions (or generalized functions) have many applications in physics and engineering. Let D denote the space of all infinitely differentiable functions with compact support. The space of all continuous linear functionals acting on the space D is called its dual space denoted by D'. A distribution f is defined as a continuous linear functional on D by

$$< f,\phi >:= \int_{-\infty}^{\infty} f(t)\phi(t)dt \quad \text{for all } \phi \in D.$$

The Dirac delta function $\delta(t)$ is defined by

$$< \delta(t-a),\phi(t) >= \int_{-\infty}^{\infty}\delta(t-a)\phi(t)dt = \phi(a) \quad (\text{for all } \phi \in D, a \in \mathbb{R}). \tag{1.102}$$

Some properties of Dirac delta function include

$$\delta(t) = 0, \quad t \neq 0,$$

$$\delta(-t) = \delta(t),$$

$$\int_{-\infty}^{\infty} \delta(t)dt = 1.$$

Every distribution has an FT and an inverse FT [49]. The Fourier transform of an arbitrary distribution in D' is not, in general, a distribution but is another linear functional which is defined over a new space. Such a functional is called an *ultradistribution* [48]. The delta function of complex argument is an ultradistribution defined as an FT of the exponential [48, p. 253] :

$$\mathcal{F}[e^{\omega t}; \tau] = \sqrt{2\pi}\delta(\tau - i\omega), \tag{1.103}$$

where $\mathcal{F}[\varphi; \tau]$ is the FT of φ. The Fourier transform of the Dirac delta function is

$$\mathcal{F}[\delta(t); \tau] := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{iy\tau} \delta(y)dy = \frac{1}{\sqrt{2\pi}}. \tag{1.104}$$

If we write a function f as a series of delta functions of complex arguments then we call it the distributional representation of the function f. For example, the gamma function has a distributional representation [6]

$$\Gamma(\sigma + i\tau) = 2\pi \sum_{m=0}^{\infty} \frac{(-1)^m}{m!} \delta(\tau - i(\sigma + m)) \quad (\sigma > 0). \tag{1.105}$$

Of course, the distributional representation makes sense only if defined as an operator acting on an element of a space of suitable test functions. It is necessary to spell out the requirements for the space of such functions. This was done in some details in [40]. A rigorous theory of distributions is given in [29] and [49].

It is possible to define the Mellin transform for all distributions belonging to the space $D'_+$of distributions on the half-line $(0, \infty)$. The procedure is to start from the space $D(0, \infty)$ of infinitely differentiable functions of compact support on $(0, \infty)$ and to consider the set Q of their Mellin transforms [26]. It can be shown that it is a space of entire functions which is isomorphic, as a linear topological space, to the space Z of Gelfand and Shilov [21]. This space can be used as a space of test functions and the one-to-one correspondence thus defined between elements of spaces $D(0, \infty)$ and Q can then be transposed to the dual spaces D'_+ and Q'. In this operation, a Mellin transform is associated with any distribution in D'_+ and the result belongs to a space Q' formed of analytic functionals.

Now, we define a space of test functions $T(\alpha_1, \alpha_2)$ containing all functions $\phi(t)$ defined on (0,∞) and with continuous derivatives of all orders going to zero as t goes to either zero or infinity. More precisely, there exist two positive numbers ζ_1, ζ_2 such that, for all integers k, the following conditions hold:

$$t^{k+1-\alpha_1-\zeta_1}\phi^{(k)}(t) \rightarrow 0, \text{as } t \rightarrow 0, \tag{1.106}$$

$$t^{k+1-\alpha_2-\zeta_2}\phi^{(k)}(t) \rightarrow 0, \text{as } t \rightarrow \infty. \tag{1.107}$$

It can be verified that all functions in $D(0, \infty)$ belong to $T(\alpha_1, \alpha_2)$. The space of distributions $T'(\alpha_1, \alpha_2)$ is then introduced as a linear space of continuous linear functionals on $T(\alpha_1, \alpha_2)$. It may be noticed that if β_1, β_2 are two real numbers such that $\alpha_1 < \beta_1 < \beta_2 < \alpha_2$, then

$$T(\beta_1, \beta_2) \subset T(\alpha_1, \alpha_2).$$

The dual spaces of distributions satisfy

$$T'(\alpha_1, \alpha_2) \subset T'(\beta_1, \beta_2).$$

More details about the construction of these spaces can be found in references [33] and [50].

The Mellin transform of a function $f \in T'(\alpha_1, \alpha_2)$ is defined by

$$M[f; s] := F_M(s) =< f, t^{s-1} > . \tag{1.108}$$

In summary, every distribution in D'_+ has a Mellin transform. The characterization of the Mellin transform of a distribution in $T(\alpha_1, \alpha_2)$ is given by the following theorems [33], [50] :

Theorem 1 [**Uniqueness**] *Let $F_M(s) = M[f, s]$ and $H_M(s) = M[h, s]$ be Mellin transforms with strips of analyticity S_f and S_h, respectively. If the strips overlap and if $F(s) = H(s)$ for $s \in S_f \cap S_{h,}$, then $f = h$ as distributions in $T^{'}(\alpha_1, \alpha_2)$, where the interval (α_1, α_2) is given by the intersection of $S_f \cap S_{h,}$ with the real axis.*

Theorem 2 *Necessary and sufficient conditions for a function $F(s)$ to be the Mellin transform of a distribution $f \in T'(\alpha_1, \alpha_2)$ are*

- *$F(s)$ is analytic in the strip $\alpha_1 < \Re(s) < \alpha_2$,*
- *For any closed substrip $\beta_1 < \Re(s) < \beta_2$ with $\alpha_1 < \beta_1 < \beta_2 < \alpha_2$ there exists a polynomial P such that $|F(s)| \leq P(|s|)$ for $\beta_1 < \Re(s) < \beta_2$.*

The Mellin transform of the Dirac delta distribution $\delta(t - t_0)$ is found by using the definition of the Dirac delta function

$$< \delta(t - t_0), \phi >= \phi(t_0),$$

we obtain

$$M[\delta(t - t_0); s] =< \delta(t - t_0), t^{s-1} >= t_0^{s-1}, \tag{1.109}$$

for any value of the positive number t_0.

CHAPTER 2

FOURIER TRANSFORM REPRESENTATION OF THE GENERALIZED HYPERGEOMETRIC FUNCTIONS

In this chapter we obtain the Fourier transform (FT) representation of the generalized hypergeometric functions ${}_pF_q[(a_p);(b_q);z]$ and consequently of the confluent and Gauss hypergeometric functions. We will apply these representations to evaluate some integrals of products of these functions. The result for Euler's gamma function is deduced as a special case.

2.1 Fourier transform representations of the generalized hypergeometric functions

To find the FT representations of the generalized hypergeometric functions, we use the integral representation (1.28). Substituting $t = e^y$ and replacing a_p by $\sigma + i\tau$ in this representation yields

$$
\begin{aligned}
&\Gamma(\sigma + i\tau)_p F_q[\sigma + i\tau, a_1, \cdots, a_{p-1}; b_1, \cdots, b_q; z] \\
&= \int_{-\infty}^{\infty} e^{iy\tau} e^{\sigma y} \exp(-e^y)_{p-1}F_q[a_1, ..., a_{p-1}; b_1, ..., b_q; ze^y] dy \\
&= \sqrt{2\pi}\mathcal{F}\left\{e^{\sigma y} \exp(-e^y)_{p-1}F_q[a_1, ..., a_{p-1}; b_1, ..., b_q; ze^y]; \tau\right\} \\
&\qquad (\sigma > 0, p \leq q + 1, |z| < 1 \text{ if } p = q + 1). \qquad (2.1)
\end{aligned}
$$

This is the Fourier transform representation of the generalized hypergeometric function, where $\mathcal{F}\{\varphi; \tau\}$ is the FT of φ defined by (1.96).

The FT representations of the CHF and GHF are given by

$$\Gamma(\sigma + i\tau)_1F_1[\sigma + i\tau; b; z] = \sqrt{2\pi}\mathcal{F}\left\{e^{\sigma y} \exp(-e^y)_0F_1[-; b; ze^y]; \tau\right\} \quad (\sigma > 0), \qquad (2.2)$$

and

$$\Gamma(\sigma + i\tau)_2F_1[\sigma + i\tau, b; c; z] = \sqrt{2\pi}\mathcal{F}\left\{e^{\sigma y} \exp(-e^y)_1F_1(b; c; ze^y); \tau\right\} \quad (\sigma > 0) \qquad (2.3)$$

respectively. By putting $z = 0$ in (2.1), the FT representation of the gamma function obtained earlier in [6] can be deduced as a special case

$$\Gamma(\sigma + i\tau) = \sqrt{2\pi}\mathcal{F}\left\{e^{\sigma y} \exp(-e^y); \tau\right\} \qquad (\sigma > 0). \qquad (2.4)$$

2.2 An integral representation of the generalized Kampé de Fériet's function

By using Parseval's identity of the FT (1.97) and the FT representation of the generalized hypergeometric functions we find that the integral of products of two generalized hypergeometric functions gives the generalized Kampé de Fériet's hypergeometric function.

Theorem 3

$$\int_{-\infty}^{\infty} \Gamma(\sigma+i\tau)_{p+1}F_q[\sigma+i\tau,(a_p);(b_q);z]\Gamma(\rho-i\tau)_{r+1}F_s[\rho-i\tau,(c_r);(d_s);w]d\tau$$
$$= \frac{\pi}{2^{\sigma+\rho-1}}\Gamma(\sigma+\rho)F^{1:p;r}_{0:q;s}\left[\begin{array}{cccc} \sigma+\rho: & (a_p); & (c_r); & \frac{z}{2},\frac{w}{2} \\ -: & (b_q); & (d_s); & \end{array}\right]$$
$$(\sigma>0, p\leq q, |z|<1 \ \textit{if}\ p=q) \ \textit{and}\ (\rho>0, r\leq s, |w|<1 \ \textit{if}\ r=s). \quad (2.5)$$

Proof. By using Parseval's identity of the FT (1.97) and the FT representation of the generalized hypergeometric functions (2.1) we have

$$\int_{-\infty}^{\infty} \Gamma(\sigma+i\tau)_{p+1}F_q[\sigma+i\tau,(a_p);(b_q);z]\Gamma(\rho-i\tau)_{r+1}F_s[\rho-i\tau,(c_r);(d_s);w]d\tau$$
$$=2\pi\int_{-\infty}^{\infty} e^{(\sigma+\rho)y}\exp\left[-2e^{y}\right]\ {}_pF_q[(a_p);(b_q);ze^{y}]_rF_s[(c_r);(d_s);we^{y}]dy. \quad (2.6)$$

For convenience, let the left-hand side of (2.6) be denoted by I. By substituting $t=2e^y$, the integral on the right-hand side of (2.6) becomes

$$I=\pi 2^{1-\sigma-\rho}\int_0^{\infty} t^{\sigma+\rho-1}e^{-t}\ {}_pF_q[(a_p);(b_q);\frac{z}{2}t]_rF_s[(c_r);(d_s);\frac{w}{2}t]dt.$$

From the series representation (1.27) of the generalized hypergeometric function ${}_pF_q$, we obtain

$$\begin{aligned}
I &= \pi 2^{1-\sigma-\rho}\int_0^\infty t^{\sigma+\rho-1}e^{-t}\ \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{\prod_{j=1}^{p}(a_j)_m}{m!\prod_{j=1}^{q}(b_j)_m}(\frac{zt}{2})^m\frac{\prod_{j=1}^{r}(c_j)_n}{n!\prod_{j=1}^{s}(d_j)_n}(\frac{wt}{2})^n dt\\
&= \pi 2^{1-\sigma-\rho}\ \sum_{n,m=0}^{\infty}\frac{\prod_{j=1}^{p}(a_j)_m}{m!\prod_{j=1}^{q}(b_j)_m}\frac{\prod_{j=1}^{r}(c_j)_n}{n!\prod_{j=1}^{s}(d_j)_n}(\frac{z}{2})^m(\frac{w}{2})^n\int_0^\infty t^{\sigma+\rho+m+n-1}e^{-t}dt\\
&= \pi 2^{1-\sigma-\rho}\ \sum_{n,m=0}^{\infty}\frac{\prod_{j=1}^{p}(a_j)_m}{m!\prod_{j=1}^{q}(b_j)_m}\frac{\prod_{j=1}^{r}(c_j)_n}{n!\prod_{j=1}^{s}(d_j)_n}(\frac{z}{2})^m(\frac{w}{2})^n\Gamma(\sigma+\rho+m+n)\\
&= \pi 2^{1-\sigma-\rho}\Gamma(\sigma+\rho)\sum_{n,m=0}^{\infty}\frac{(\sigma+\rho)_{m+n}}{n!m!}\frac{\prod_{j=1}^{r}(c_j)_n}{\prod_{j=1}^{s}(d_j)_n}\frac{\prod_{j=1}^{p}(a_j)_m}{\prod_{j=1}^{q}(b_j)_m}(\frac{w}{2})^n(\frac{z}{2})^m. \qquad (2.7)
\end{aligned}$$

In view of the definition of the generalized Kampé de Fériet's hypergeometric function (1.42), the final series in (2.7) is absolutely convergent for

$$(\sigma>0, p\le q, |z|<1 \text{ if } p=q) \text{ and } (\rho>0, r\le s, |w|<1 \text{ if } r=s).$$

The term by term integration is justified by a corollary of dominated convergence theorem [20, p. 55, Theorem 2.25]. So we have (2.5). ■

Corollary 4 *The generalized hypergeometric function satisfies the following in-*

tegral equation

$$\int_{-\infty}^{\infty} \Gamma(\sigma+i\tau)_{p+1}F_q(\sigma+i\tau, a_1, \cdots, a_p; b_1, \cdots, b_q; z)\Gamma(\rho-\tau i)d\tau$$
$$= \pi 2^{1-\sigma-\rho}\Gamma(\sigma+\rho)_{p+1}F_q(\sigma+\rho, a_1, \cdots, a_p; b_1, \cdots, b_q; \frac{z}{2})$$
$$(\sigma, \rho > 0, p \leq q, |z| < 1 \text{ if } p = q). \tag{2.8}$$

Proof. The proof of (2.8) follows from (2.5) when we take $w = 0$. ■

Corollary 5

$$\int_{-\infty}^{\infty} |\Gamma(\sigma+i\tau)|^2 {}_{p+1}F_q[\sigma+i\tau, a_1, \cdots, a_p; b_1, \cdots, b_q; z]d\tau$$
$$= \pi 2^{1-2\sigma}\Gamma(2\sigma)_{p+1}F_q[2\sigma, a_1, \cdots, a_p; b_1, \cdots, b_q; \frac{z}{2}]$$
$$(\sigma > 0, p \leq q, |z| < 1 \text{ if } p = q). \tag{2.9}$$

Proof. Taking $\sigma = \rho$ in (2.8) we get (2.9). ■

The following integral identities for CHFs and GHFs are consequences of the above theorem:

$$\begin{aligned} & \int_{-\infty}^{\infty} \Gamma(\sigma+i\tau)_1F_1[\sigma+i\tau; b; z]\Gamma(\rho-\tau i)d\tau \\ = & \frac{\pi}{2^{\sigma+\rho-1}}\Gamma(\sigma+\rho)_1F_1[\sigma+\rho; b; \frac{z}{2}] \ (\sigma, \rho > 0), \end{aligned} \tag{2.10}$$

$$\int_{-\infty}^{\infty} |\Gamma(\sigma+i\tau)|^2 {}_1F_1[\sigma+i\tau; b; z]d\tau = \frac{\pi}{2^{2\sigma-1}}\Gamma(2\sigma)_1F_1[2\sigma; b; \frac{z}{2}] \ (\sigma > 0), \tag{2.11}$$

$$\begin{aligned} & \int_{-\infty}^{\infty} \Gamma(\sigma+i\tau)_2F_1[\sigma+i\tau; b; c; z]\Gamma(\rho-i\tau)d\tau \\ = & \frac{\pi}{2^{\sigma+\rho-1}}\Gamma(\sigma+\rho)_2F_1[\sigma+\rho; b; c; \frac{z}{2}] \ (\sigma, \rho > 0, |z| < 1), \end{aligned} \tag{2.12}$$

$$
\begin{aligned}
&\int_{-\infty}^{\infty} |\Gamma(\sigma+i\tau)|^2 \, {}_2F_1[\sigma+i\tau; b; c; z] d\tau \\
= \quad & \frac{\pi}{2^{2\sigma-1}} \Gamma(2\sigma) \, {}_2F_1[2\sigma; b; c; \frac{z}{2}] \quad (\sigma > 0, |z| < 1). \qquad (2.13)
\end{aligned}
$$

Taking $z = 0$, we get the well-known "norm squared" of Γ (see, for example, [34, p.143, (5.13.2)]:

$$
\int_{-\infty}^{\infty} |\Gamma(\sigma+i\tau)|^2 \, d\tau = \pi 2^{1-2\sigma} \Gamma(2\sigma) \quad (\sigma > 0). \qquad (2.14)
$$

2.3 Some applications to the confluent and Gauss hypergeometric functions

In this section we apply the main result (2.5) to find some integral identities for CHFs and GHFs.

Theorem 6 *The CHFs satisfy the following identity:*

$$
\begin{aligned}
&\int_{-\infty}^{\infty} \Gamma(\sigma+i\tau) {}_1F_1[\sigma+i\tau; b; z] \Gamma(\rho-i\tau) {}_1F_1[\rho-i\tau; c; z] d\tau \\
&= \pi 2^{1-\sigma-\rho} \Gamma(\sigma+\rho) {}_3F_3[\sigma+\rho, \frac{b+c-1}{2}, \frac{b+c}{2}; b, c, b+c-1; 2z] \quad (\sigma, \rho > 0).
\end{aligned}
\qquad (2.15)
$$

Proof. Putting $p = r = 0$, $q = s = 1$ and $w = z$ in (2.5), we find

$$
\begin{aligned}
&\int_{-\infty}^{\infty} \Gamma(\sigma+i\tau) {}_1F_1[\sigma+i\tau; b; z] \Gamma(\rho-i\tau) {}_1F_1[\rho-i\tau; c; z] d\tau \\
&= \frac{\pi}{2^{\sigma+\rho-1}} \Gamma(\sigma+\rho) F^{1:0;0}_{0:1;1} \left[\begin{array}{cccc} \sigma+\rho: & -; & -; & \frac{z}{2}, \frac{z}{2} \\ -: & b; & c; & \end{array} \right] (\sigma, \rho > 0). \qquad (2.16)
\end{aligned}
$$

Now, using the reduction formula [4, (3.2)]

$$F^{1:0;0}_{0:1;1}\left[\begin{array}{cccc} \alpha: & -; & -; & \\ & & & x,x \\ -: & \nu; & \sigma; & \end{array}\right] = {}_3F_3[\alpha, \frac{\nu+\sigma-1}{2}, \frac{\nu+\sigma}{2}; \nu, \sigma, \nu+\sigma-1; 4x],$$

we get (2.15). ■

Corollary 7 *The following identities, involving the CHFs, hold:*

$$\int_{-\infty}^{\infty} \Gamma(\sigma+i\tau)_1F_1[\sigma+i\tau;b;z]\Gamma(\rho-i\tau)_1F_1[\rho-i\tau;b;z]d\tau$$
$$= \pi 2^{1-\sigma-\rho}\Gamma(\sigma+\rho)_2F_2[\sigma+\rho, b-\frac{1}{2}; b, 2b-1; 2z] \quad (\sigma,\rho>0), \tag{2.17}$$

$$\int_{-\infty}^{\infty} \left|\Gamma(\sigma+i\tau)_1F_1[\sigma+i\tau;b;z]\right|^2 d\tau$$
$$= \pi 2^{1-2\sigma}\Gamma(2\sigma)_2F_2[2\sigma, b-\frac{1}{2}; b, 2b-1; 2z] \quad (\sigma>0). \tag{2.18}$$

Proof. By taking $b=c$ in (2.15) we get (2.17), and as a special case, taking $\sigma=\rho$, we get (2.18). ■

Theorem 8 *The CHFs satisfy the following identity:*

$$\int_{-\infty}^{\infty} \Gamma(\sigma+i\tau)_1F_1[\sigma+i\tau;b;z]\Gamma(\rho-i\tau)_1F_1[\rho-i\tau;b;-z]d\tau$$
$$= \pi 2^{1-\sigma-\rho}\Gamma(\sigma+\rho)_2F_3[\frac{\sigma+\rho}{2}, \frac{\sigma+\rho+1}{2}; \frac{b}{2}, \frac{b+1}{2}, b; -\frac{z^2}{4}] \quad (\sigma,\rho>0). \tag{2.19}$$

Proof. Putting $p=r=0$, $q=s=1, b=c$ and $w=-z$ in (2.5), and using the reduction formula [4, (3.7)]

$$F^{1:0;0}_{0:1;1}\left[\begin{array}{cccc} \alpha: & -; & -; & \\ & & & x,-x \\ -: & \nu; & \nu; & \end{array}\right] = {}_2F_3[\frac{\alpha}{2}, \frac{\alpha+1}{2}; \frac{\nu}{2}, \frac{\nu+1}{2}, \nu; -x^2],$$

we obtain the result (2.19). ■

We obtain similar results for the GHFs.

Theorem 9 *The GHFs satisfy the following identities:*

$$\int_{-\infty}^{\infty} \Gamma(\sigma+i\tau)_2F_1[\sigma+i\tau,b;c;z]\Gamma(\rho-i\tau)_2F_1[\rho-i\tau,b;c;-z]d\tau$$
$$= \pi 2^{1-\sigma-\rho}\Gamma(\sigma+\rho)_4F_3[\frac{\sigma+\rho}{2},\frac{\sigma+\rho+1}{2},b,c-b;\frac{c}{2},\frac{c+1}{2},c;\frac{z^2}{4}]$$
$$(\sigma,\rho>0,|z|<1,\Re(c)>\Re(b)>0>0) \tag{2.20}$$

and

$$\int_{-\infty}^{\infty} \Gamma(\sigma+i\tau)_2F_1[\sigma+i\tau,b;2b;z]\Gamma(\rho-i\tau)_2F_1[\rho-i\tau,c;2c;-z]d\tau$$
$$= \pi 2^{1-\sigma-\rho}\Gamma(\sigma+\rho)_4F_3\left[\frac{\sigma+\rho}{2},\frac{\sigma+\rho+1}{2},\frac{b+c}{2},\frac{b+c+1}{2};b+\frac{1}{2},c+\frac{1}{2},b+c;\frac{z^2}{4}\right]$$
$$(\sigma,\rho>0,|z|<1). \tag{2.21}$$

Proof. By using a similar procedure and using the following reduction formulas [4, (3.4), (3.5)]:

$$F^{1:1;1}_{0:1;1}\left[\begin{array}{ccc} \alpha: & \lambda; & \lambda; \\ -: & \nu; & \nu; \end{array} x,-x\right] = {}_4F_3[\frac{\alpha}{2},\frac{\alpha+1}{2},\lambda,\lambda-\nu;\nu,\frac{\nu}{2},\frac{\nu+1}{2};x^2],$$

$$F^{1:1;1}_{0:1;1}\left[\begin{array}{ccc} \alpha: & \lambda; & \mu; \\ -: & 2\lambda; & 2\mu; \end{array} x,-x\right]$$
$$= {}_4F_3\left[\frac{\alpha}{2},\frac{\alpha+1}{2},\frac{\lambda+\mu}{2},\frac{\lambda+\mu+1}{2};\lambda+\frac{1}{2},\mu+\frac{1}{2},\lambda+\mu;x^2\right],$$

we obtain (2.20) and (2.21). ■

We can get more integral formulae by different choices of the parameters a, b and c.

In this chapter, we have obtained the Fourier transform representation of the generalized hypergeometric functions. By using this representation we have found that the integral of products of two generalized hypergeometric functions is a generalized Kampé de Fériet's hypergeometric function.

It is worth mentioning here that, whenever a generalized hypergeometric function reduces to the confluent or Gauss hypergeometric functions and other hypergeometric functions, the results become more important for applications. Most of the special functions of mathematical physics and engineering, such as the Bessel, Laguerre and Legendre functions, can be expressed in terms of the confluent or Gauss hypergeometric functions. Therefore, the main result in this chapter can play an important role in the theory of special functions of applied mathematics and mathematical physics.

We conclude this chapter by remarking that several formulae can be obtained from the main result by appropriate choices of the parameters.

CHAPTER 3

DISTRIBUTIONAL REPRESENTATION OF THE GENERALIZED HYPERGEOMETRIC FUNCTIONS

In the previous chapter we used the FT representation of the generalized hypergeometric function to evaluate integrals of products of two generalized hypergeometric functions. In this chapter we obtain the distributional representation of the generalized hypergeometric functions and consequently of confluent and Gauss hypergeometric functions. We use this representation to evaluate integrals involving generalized hypergeometric, confluent and Gauss hypergeometric functions.

3.1 Distributional representation of the generalized hypergeometric functions

In this section we obtain the distributional representations of the generalized hypergeometric functions. Recall the FT representation of the generalized hypergeometric function obtained in the previous chapter

$$\Gamma(\sigma+i\tau)_pF_q[\sigma+i\tau,a_1,...,a_{p-1};b_1,...,b_q;z]$$
$$=\sqrt{2\pi}\mathcal{F}\left\{e^{\sigma y}\exp(-e^y)_{p-1}F_q[a_1,...,a_{p-1};b_1,...,b_q;ze^y];\tau\right\}$$
$$(\sigma>0,p\leq q+1). \tag{3.1}$$

Theorem 10 *The generalized hypergeometric function has a distributional representation*

$$\Gamma(\sigma+i\tau)_pF_q[\sigma+i\tau,a_1,...,a_{p-1};b_1,...,b_q;z]$$
$$=2\pi\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{(-1)^m z^n(a_1)_n\cdots(a_{p-1})_n}{m!n!(b_1)_n\cdots(b_q)_n}\delta(\tau-i(\sigma+n+m))$$
$$(\sigma>0,p\leq q+1). \tag{3.2}$$

Proof. Using the series expansion

$$e^{\sigma y}\exp(-e^y)_{p-1}F_q[a_1,...,a_{p-1};b_1,...,b_q;ze^y]$$
$$=\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{(-1)^m z^n(a_1)_n\cdots(a_{p-1})_n}{m!n!(b_1)_n\cdots(b_q)_n}e^{(\sigma+m+n)y}, \tag{3.3}$$

and from (1.103), the FT of the exponential gives a delta function, we can rewrite (3.1) as a series of delta functions, namely (3.2) ■

Let Ω be the space of all entire functions ϕ for which, for fixed z, the series

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{(-1)^m}{m!}\frac{z^n(a_1)_n\cdots(a_{p-1})_n}{n!(b_1)_n\cdots(b_q)_n}\phi(w+n+m)\ (p\leq q+1),$$

converges for all w. We call Ω the space of test functions and its members the test functions. It is to be noted that Ω is not empty as e^w is in Ω.

The representation in (3.2) is only meaningful when defined as the inner product with a test function that belongs to Ω.

The FT and the distributional representations of the confluent hypergeometric function (CHF), for $\sigma>0$, are given by

$$\Gamma(\sigma+i\tau)_1F_1[\sigma+i\tau;b;z]=\sqrt{2\pi}\mathcal{F}\left\{e^{\sigma y}\exp(-e^y)_0F_1(-;b;ze^y);\tau\right\} \tag{3.4}$$

and

$$\Gamma(\sigma+i\tau)_1F_1[\sigma+i\tau;b;z]=2\pi\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{(-1)^m}{m!}\frac{(z)^n}{n!(b)_n}\delta(\tau-i(\sigma+n+m)) \tag{3.5}$$

respectively.

Similarly, we obtain the FT and the distributional representations of the Gauss hypergeometric function (GHF), for $\sigma>0$:

$$\Gamma(\sigma+i\tau)_2F_1[\sigma+i\tau,b;c;z]=\sqrt{2\pi}\mathcal{F}\left\{e^{\sigma y}\exp(-e^y)_1F_1(b;c;ze^y);\tau\right\}, \tag{3.6}$$

$$\Gamma(\sigma+i\tau)_2F_1[\sigma+i\tau,b;c;z]=2\pi\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{(-1)^m}{m!}\frac{(z)^n(b)_n}{n!(c)_n}\delta(\tau-i(\sigma+n+m)). \tag{3.7}$$

By putting $z=0$ in (2.1) and (3.2), the FT and the distributional representations of the gamma function obtained earlier in [6] can be deduced as special cases.

3.2 Some applications of the distributional representation

The distributional representation of the generalized hypergeometric functions is obtained as a series of delta functions, which is convergent in the distributional sense if its inner product with any test function converges.

Consider the inner product of the generalized hypergeometric function with a good function $\phi(s)(s = \rho + i\tau)$. Then (3.2) yields

$$\begin{aligned}
&< \Gamma(\sigma + i\tau)_p F_q[\sigma + i\tau, a_1, ..., a_{p-1}; b_1, ..., b_q; z], \phi(\rho + i\tau) > \\
&= 2\pi \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{(-1)^m}{m!} \frac{(z)^n (a_1)_n \cdots (a_{p-1})_n}{n!(b_1)_n \cdots (b_q)_n} \phi(\rho - \sigma - n - m) \\
&\qquad (\sigma > 0, p \leq q + 1), \qquad (3.8)
\end{aligned}$$

where we used the definition of the Dirac-delta function and its linearity property. This inner product is well defined for all those functions for which these infinite series converge. Now take the inner product of the generalized hypergeometric function with the set of functions $\left\{y^{(\rho+i\tau)u}\right\}_{u\in\mathbb{C}}$ $(y > 0)$, we get

$$\begin{aligned}
&\int_{-\infty}^{\infty} \Gamma(\sigma + i\tau)_p F_q[\sigma + i\tau, (a_{p-1}); (b_q); z] y^{(\rho+i\tau)u} d\tau \\
=\ & 2\pi y^{(\rho-\sigma)u} \exp(-y^{-u})_{p-1} F_q[(a_{p-1}); (b_q); zy^{-u}]\ (\sigma > 0, p \leq q + 1) \qquad (3.9)
\end{aligned}$$

By using the distributional representation of the gamma function (1.105), and replacing y^{-u} by w we get from (3.9) the known generating function (see [38, p.141, eq. (19)])

$$\begin{aligned}
&\sum_{m=0}^{\infty} \frac{(-1)^m}{m!}\ {}_p F_q[-m, (a_{p-1}); (b_q); z] w^m \\
=\ & \exp(-w)_{p-1} F_q[(a_{p-1}); (b_q); zw]\ (p \leq q + 1). \qquad (3.10)
\end{aligned}$$

Now, taking $u = 0$ in (3.9) gives

$$\begin{aligned}&\int_{-\infty}^{\infty} \Gamma(\sigma + i\tau)_p F_q[\sigma + i\tau, (a_{p-1}); (b_q); z] d\tau \\ = \quad & 2\pi \exp(-1)_{p-1} F_q[(a_{p-1}); (b_q); z] \quad (\sigma > 0).\end{aligned} \tag{3.11}$$

By taking specific values for p and q, we get new formulas for CHF and GHF. By letting $p = q = 1$ in (3.8), (3.9) and (3.11), one obtains the following results for the CHF

$$\begin{aligned}&\int_{-\infty}^{\infty} \Gamma(\sigma + i\tau)_1 F_1[\sigma + i\tau; b; z] \phi(\rho + i\tau) d\tau \\ = \quad & 2\pi \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{(-1)^m}{m!} \frac{(z)^n}{n!(b)_n} \phi(\rho - \sigma - n - m) \quad (\sigma > 0),\end{aligned} \tag{3.12}$$

$$\begin{aligned}&\int_{-\infty}^{\infty} \Gamma(\sigma + i\tau)_1 F_1[\sigma + i\tau; b; z] y^{(\rho+i\tau)u} d\tau \\ = \quad & 2\pi y^{(\rho-\sigma)u} \exp(-y^{-u})_0 F_1[-; b; zy^{-u}] \quad (\sigma > 0),\end{aligned} \tag{3.13}$$

and

$$\begin{aligned}&\int_{-\infty}^{\infty} \Gamma(\sigma + i\tau)_1 F_1[\sigma + i\tau; b; z] d\tau \\ = \quad & 2\pi \exp(-1)_0 F_1[-; b; z] \quad (\sigma > 0).\end{aligned} \tag{3.14}$$

Similarly we get the following results for the GHF

$$\begin{aligned}&\int_{-\infty}^{\infty} \Gamma(\sigma + i\tau)_2 F_1[\sigma + i\tau, b; c; z] \phi(\rho + i\tau) d\tau \\ = \quad & 2\pi \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{(-1)^m}{m!} \frac{(z)^n (b)_n}{n!(c)_n} \phi(\rho - \sigma - n - m) \quad (\sigma > 0),\end{aligned} \tag{3.15}$$

$$\begin{aligned}&\int_{-\infty}^{\infty}\Gamma(\sigma+i\tau)_2F_1[\sigma+i\tau,b;c;z]y^{(\rho+i\tau)u}d\tau\\ =\;& 2\pi y^{(\rho-\sigma)u}\exp(-y^{-u})_1F_1[b;c;zy^{-u}]\quad(\sigma>0),\end{aligned}\tag{3.16}$$

and

$$\begin{aligned}&\int_{-\infty}^{\infty}\Gamma(\sigma+i\tau)_2F_1[\sigma+i\tau,b;c;z]d\tau\\ =\;& 2\pi\exp(-1)_1F_1[b;c;z]\quad(\sigma>0).\end{aligned}\tag{3.17}$$

To prove that the results obtained by using the distributional representation are consistent with the results obtained in the previous chapter by using FT representation, consider the following integral

$$I=\int_{-\infty}^{\infty}\Gamma(\sigma+i\tau)_{p+1}F_q[\sigma+i\tau,(a_p);(b_q);z]\Gamma(\rho-i\tau)_{r+1}F_s[\rho-i\tau,(c_r);(d_s);w]d\tau.$$

Using (3.8) we have

$$I=2\pi\sum_{n,m=0}^{\infty}\frac{(-1)^m z^n\prod_{j=1}^{p}(a_j)_n}{m!n!\prod_{j=1}^{q}(b_j)_n}\Gamma(\rho+\sigma+n+m)_{r+1}F_s[\rho+\sigma+n+m,(c_r);(d_s);w].\tag{3.18}$$

Replacing ${}_{r+1}F_s$ by its series representation and interchanging the order of summation, we have

$$I=2\pi\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\frac{z^n\prod_{j=1}^{p}(a_j)_n}{n!\prod_{j=1}^{q}(b_j)_n}\frac{w^k\prod_{j=1}^{r}(c_j)_k}{k!\prod_{j=1}^{s}(d_j)_k}\sum_{m=0}^{\infty}\frac{(-1)^m}{m!}\Gamma(\rho+\sigma+n+m+k).$$

Interchanging the order of summations is justified by the absolute convergence of the series for

$$(\sigma>0,p\le q,|z|<1\text{ if }p=q)\text{ and }(\rho>0,r\le s,|w|<1\text{ if }r=s).$$

Replacing $\sum_{m=0}^{\infty}\frac{(-1)^m}{m!}\Gamma(\rho+\sigma+n+m+k)$ by $\Gamma(\rho+\sigma+n+k)(\frac{1}{2})^{\rho+\sigma+n+k}$ and using the definition of generalized Kampé de Fériet's hypergeometric function (1.42), we get

$$\int_{-\infty}^{\infty} \Gamma(\sigma+i\tau)_{p+1}F_q[\sigma+i\tau,(a_p);(b_q);z]\Gamma(\rho-i\tau)_{r+1}F_s[\rho-i\tau,(c_r);(d_s);w]d\tau$$

$$= \frac{\pi}{2^{\sigma+\rho-1}}\Gamma(\sigma+\rho)F^{1:p;r}_{0:q;s}\left[\begin{array}{rccc} \sigma+\rho: & (a_p); & (c_r); & \frac{z}{2},\frac{w}{2} \\ -: & (b_q); & (d_s); & \end{array}\right]$$

$$(\sigma>0, p\leq q, |z|<1 \text{ if } p=q) \text{ and } (\rho>0, r\leq s, |w|<1 \text{ if } r=s). \qquad (3.19)$$

3.3 Some applications to the confluent and Gauss hypergeometric functions

In this section we use the distributional representation of the generalized hypergeometric and the distributional representations for CHFs and GHFs to find some integral identities.

Theorem 11 *The following identity holds :*

$$\int_{-\infty}^{\infty} \Gamma(\sigma+i\tau)_pF_q[\sigma+i\tau,(a_{p-1});(b_q);z]\Gamma(c-\rho-i\tau)_1F_1[\rho+i\tau;c;w]d\tau$$

$$= \pi e^w\frac{\Gamma(c-\rho+\sigma)}{2^{c-\rho+\sigma-1}}F^{1:p-1;0}_{0:q;1}\left[\begin{array}{rccc} c-\rho+\sigma: & (a_{p-1}); & -; & \frac{z}{2},-\frac{w}{2} \\ -: & (b_q); & c; & \end{array}\right]$$

$$(\sigma>0, c>\rho, p\leq q, |z|<1 \ \textit{if}\ p=q\ ). \qquad (3.20)$$

Proof. By using (3.8) we have

$$\int_{-\infty}^{\infty}\Gamma(\sigma+i\tau)_pF_q[\sigma+i\tau,(a_{p-1});(b_q);z]\Gamma(c-\rho-i\tau)_1F_1[\rho+i\tau;c;w]d\tau$$

$$=2\pi\sum_{n,m=0}^{\infty}\frac{(-1)^m}{m!}\frac{z^n\prod_{j=1}^{p-1}(a_j)_n}{n!\prod_{j=1}^{q}(b_j)_n}\Gamma(c-\rho+\sigma+n+m)_1F_1[\rho-\sigma-n-m;c;w]$$

$$(\sigma>0,p\leq q+1,|z|<1 \text{ if } p=q\ ). \tag{3.21}$$

For convenience, let the left-hand side of (3.21) be denoted by I. By using Kummer's transformation formula for confluent hypergeometric function [3, p. 253]

$${}_1F_1(a;c;z)=e^z_1F_1(c-a;c;-z),$$

the right hand side of (3.21) becomes

$$I=2\pi\sum_{n,m=0}^{\infty}\frac{(-1)^m}{m!}\frac{z^n\prod_{j=1}^{p-1}(a_j)_n}{n!\prod_{j=1}^{q}(b_j)_n}e^w\Gamma(c-\rho+\sigma+n+m)_1F_1[c-\rho+\sigma+n+m;c;-w].$$

Now, replacing ${}_1F_1(c-\rho+\sigma+n+m;c;-w)$ by its series representation and interchanging the order of summation, we get

$$I=2\pi e^w\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\frac{z^n\prod_{j=1}^{p-1}(a_j)_n}{n!\prod_{j=1}^{q}(b_j)_n}\frac{(-w)^k}{k!(c)_k}\sum_{m=0}^{\infty}\frac{(-1)^m\Gamma(c-\rho+\sigma+n+m+k)}{m!}. \tag{3.22}$$

Replace $\sum_{m=0}^{\infty}\frac{(-1)^m\Gamma(c-\rho+\sigma+n+m+k)}{m!}$ by

$$\Gamma(c-\rho+\sigma+n+k)_0F_1[-,c-\rho+\sigma+n+k;-1]=\Gamma(c-\rho+\sigma+n+k)(\frac{1}{2})^{c-\rho+\sigma+n+k}.$$

Thus we have

$$I=\pi e^w\frac{\Gamma(c-\rho+\sigma)}{2^{c-\rho+\sigma-1}}\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\frac{(\frac{z}{2})^n\prod_{j=1}^{p-1}(a_j)_n}{n!\prod_{j=1}^{q}(b_j)_n}\frac{(-\frac{w}{2})^k}{k!(c)_k}(c-\rho+\sigma)_{n+k}.$$

By using the definition of generalized Kampé de Fériet's hypergeometric function (1.42), we obtain (3.20). ■

Theorem 12 *The following identities involving the CHFs hold:*

$$\int_{-\infty}^{\infty} \Gamma(\sigma+i\tau)_1F_1[\sigma+i\tau;b;z]\Gamma(c-\rho-i\tau)_1F_1[\rho+i\tau;b;z]d\tau$$
$$=\pi\frac{e^z}{2^{c-\rho+\sigma-1}}\Gamma(c-\rho+\sigma)_2F_3\left[\frac{c-\rho+\sigma}{2},\frac{c-\rho+\sigma+1}{2};\frac{b}{2},\frac{b+1}{2},b;-\frac{z^2}{4}\right]$$
$$(\sigma>0,\Re(c)>\rho>0), \tag{3.23}$$

$$\int_{-\infty}^{\infty} \Gamma(\sigma+i\tau)_1F_1[\sigma+i\tau;b;z]\Gamma(c-\rho-i\tau)_1F_1[\rho+i\tau;c;-z]d\tau$$
$$=\pi\frac{e^{-z}}{2^{c-\rho+\sigma-1}}\Gamma(c-\rho+\sigma)_3F_3\left[c-\rho+\sigma,\frac{b+c-1}{2},\frac{b+c}{2};b,c,b+c-1;2z\right]$$
$$(\sigma>0,\Re(c)>\rho>0). \tag{3.24}$$

Proof. Putting $p=q=1, b=c$ and $w=z$ in (3.20), and using the reduction formula [4, (3.7)]

$$F^{1:0;0}_{0:1;1}\left[\begin{array}{ccc} \alpha: & -; & -; \\ -: & \nu; & \nu; \end{array} x,-x\right] = {}_2F_3\left[\frac{\alpha}{2},\frac{\alpha+1}{2};\frac{\nu}{2},\frac{\nu+1}{2},\nu;-x^2\right],$$

we obtain the result (3.23). We get (3.24) by taking $p=q=1$ and $w=-z$ in (3.20), and using the reduction formula [4, (3.2)]

$$F^{1:0;0}_{0:1;1}\left[\begin{array}{ccc} \alpha: & -; & -; \\ -: & \nu; & \sigma; \end{array} x,x\right] = {}_3F_3\left[\alpha,\frac{\nu+\sigma-1}{2},\frac{\nu+\sigma}{2};\nu,\sigma,\nu+\sigma-1;4x\right].$$

■

By carrying out a similar procedure and employing Euler's transformation of the hypergeometric function [18, p. 64]

$$ {}_2F_1[a, b; c; z] = (1-z)^{-b}\ {}_2F_1[c-a, b; c; \frac{z}{z-1}], $$

we find similar new results for GHFs.

Theorem 13 *The GHFs satisfy the following identity*

$$ \int_{-\infty}^{\infty} \Gamma(\sigma+i\tau)_pF_q[\sigma+i\tau, (a_{p-1}); (b_q); z]\Gamma(h-\rho-i\tau)_2F_1[\rho+i\tau, d; h; w]d\tau $$

$$ = \pi(1-w)^{-d}\frac{\Gamma(c-\rho+\sigma)}{2^{c-\rho+\sigma-1}}F_{0:q;1}^{1:p-1;1}\left[\begin{array}{rcc} h-\rho+\sigma: & (a_{p-1}); & d; \\ -: & (b_q); & h; \end{array} \frac{z}{2}, \frac{w}{2(w-1)}\right] $$

$$ (h > \rho > 0, \sigma > 0, |w| < \frac{1}{2}, p \le q, |z| < 1 \text{ } \textit{if } p = q). \tag{3.25} $$

3.4 Applications to the Riemann zeta function

In this section we consider the action of the generalized hypergeometric function on the Riemann zeta function in the critical strip. Letting $\phi(s) = \zeta(s)$ in (3.8) yields

$$ \int_{-\infty}^{\infty} \Gamma(\sigma+i\tau)_pF_q[\sigma+i\tau, a_1, \cdots, a_{p-1}; b_1, \cdots, b_q; z]\zeta(\rho+i\tau)d\tau $$

$$ = 2\pi\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{(z)^n(-1)^m(a_1)_n\cdots(a_{p-1})_n}{n!m!(b_1)_n\cdots(b_q)_n}\zeta(\rho-\sigma-n-m) $$

$$ (0 < \rho < 1, \sigma > 0, p \le q+1), \tag{3.26} $$

and, consequently, for CHF and GHF

$$\begin{aligned}&\int_{-\infty}^{\infty}\Gamma(\sigma+i\tau)_1F_1[\sigma+i\tau;b;z]\zeta(\rho+i\tau)d\tau\\ = \quad & 2\pi\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{(z)^n(-1)^m}{n!m!(b)_n}\zeta(\rho-\sigma-n-m) \quad (0<\rho<1,\sigma>), \end{aligned} \tag{3.27}$$

$$\begin{aligned}&\int_{-\infty}^{\infty}\Gamma(\sigma+i\tau)_2F_1[\sigma+i\tau,b;c;z]\zeta(\rho+i\tau)d\tau\\ &=2\pi\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{(z)^n(-1)^m(b)_n}{n!m!(c)_n}\zeta(\rho-\sigma-n-m)\\ &\qquad (0<\rho<1,\sigma>0)\end{aligned} \tag{3.28}$$

provided that the series is convergent. Taking $z=0$ yields

$$\begin{aligned}&\int_{-\infty}^{\infty}\Gamma(\sigma+i\tau)\zeta(\rho+i\tau)d\tau\\ = \quad & 2\pi\sum_{m=0}^{\infty}\frac{(-1)^m}{m!}\zeta(\rho-\sigma-m) \quad (0<\rho<1,\sigma>0).\end{aligned} \tag{3.29}$$

A special case of (3.29) $\rho=\sigma$ leads to the formula obtained earlier in [41, (4.8)]

$$\int_{-\infty}^{\infty}\Gamma(\sigma+i\tau)\zeta(\sigma+i\tau)d\tau=2\pi\sum_{m=0}^{\infty}\frac{(-1)^m}{m!}\zeta(-m) \quad (0<\sigma<1). \tag{3.30}$$

Now, in (3.28) letting $z=1$ gives

$$\begin{aligned}&\int_{-\infty}^{\infty}\Gamma(\sigma+i\tau)_2F_1[\sigma+i\tau,b;c;1]\zeta(\rho+i\tau)d\tau\\ &=2\pi\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{(1)^n(-1)^m(b)_n}{n!m!(c)_n}\zeta(\rho-\sigma-n-m)\\ &\qquad (0<\rho<1,\sigma>0),\end{aligned} \tag{3.31}$$

on the other hand, using the distributional representation of the gamma function (1.105), we get

$$\int_{-\infty}^{\infty} \Gamma(\sigma+i\tau)_2F_1[\sigma+i\tau,b;c;1]\zeta(\rho+i\tau)d\tau$$
$$= 2\pi \sum_{m=0}^{\infty} \frac{(-1)^m}{m!}\zeta(\rho-\sigma-m)_2F_1[-m,b;c;1]$$
$$(0<\rho<1, \sigma>0). \tag{3.32}$$

By using the formula [18, p.61]

$$_2F_1[a,b;c;1] = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} \quad (\Re(c)>\Re(b)>0, \Re(c-a-b)>0), \tag{3.33}$$

$$_2F_1[-n,b;c;1] = \frac{(c-b)_n}{(c)_n} \quad (\Re(c)>\Re(b)>0),$$

we have from (3.32)

$$\frac{\Gamma(c)}{\Gamma(c-b)}\int_{-\infty}^{\infty} \Gamma(\sigma+i\tau)\frac{\Gamma(c-b-\sigma-i\tau)}{\Gamma(c-\sigma-i\tau)}\zeta(\rho+i\tau)d\tau$$
$$= 2\pi \sum_{m=0}^{\infty} \frac{(-1)^m}{m!}\frac{(c-b)_m}{(c)_m}\zeta(\rho-\sigma-m)$$
$$(0<\rho<1, \sigma>0, (\Re(c)>\Re(b)>0)), \tag{3.34}$$

and special case $\rho=\sigma$ leads to

$$\frac{\Gamma(c)}{\Gamma(c-b)}\int_{-\infty}^{\infty} \Gamma(\sigma+i\tau)\frac{\Gamma(c-b-\sigma-i\tau)}{\Gamma(c-\sigma-i\tau)}\zeta(\sigma+i\tau)d\tau$$
$$= 2\pi \sum_{m=0}^{\infty} \frac{(-1)^m}{m!}\frac{(c-b)_m}{(c)_m}\zeta(-m)$$
$$(0<\sigma<1, \Re(c)>\Re(b)>0, \Re(c-a-b)>0). \tag{3.35}$$

3.5 Distributional representation of a Mellin transformable function

In this section we obtain the distributional representation for a given Mellin transformable function. Applications of this representation are used to evaluate the integrals of products of a Mellin transform of a given function with other functions. The distributional representation of Euler's gamma function is deduced as a special case. Ramanujan's master theorem is deduced from this representation also as a special case.

Theorem 14 *Let f be a Mellin transformable function which has a Laurent expansion about zero*

$$f(x) = \sum_{n=-\infty}^{\infty} a_n x^n.$$

Then, its Mellin transform $M[f; \sigma + i\tau] = F_M(\sigma + i\tau)$ has a distributional representation

$$F_M(\sigma + i\tau) = 2\pi \sum_{n=-\infty}^{\infty} a_n \delta(\tau - i(\sigma + n)) \quad (\sigma \in S_f) \tag{3.36}$$

where σ belongs to the strip of definition of the Mellin transform of f.

Proof. Putting $t = e^x$, the Mellin transform can be represented as

$$F_M(\sigma + i\tau) = \int_{-\infty}^{\infty} e^{ix\tau} e^{\sigma x} f(e^x) dx = \sqrt{2\pi} \mathcal{F}\left[e^{\sigma x} f(e^x); \tau\right], \tag{3.37}$$

where $\mathcal{F}[\varphi; \tau]$ is the FT of φ. Using the Laurent expansion of $f(x)$

$$e^{\sigma x} f(e^x) = \sum_{n=-\infty}^{\infty} a_n e^{(\sigma+n)x},$$

as the Fourier transform of the exponential gives a delta function

$$\mathcal{F}[e^{\omega t}; \tau] = \sqrt{2\pi}\delta(\tau - i\omega), \tag{3.38}$$

we can rewrite (3.37) as a series of delta functions

$$F_M(\sigma + i\tau) = 2\pi \sum_{n=-\infty}^{\infty} a_n \delta(\tau - i(\sigma + n)).$$

This proves that every Mellin transformable function has a distributional representation. ■

This representation is convergent in the distributional sense if its inner product with all possible test function converges in the space of test functions. If we take $f(t) = e^{-t}$, we get the distributional representation for Euler's gamma function obtained in [6]

$$\Gamma(\sigma + i\tau) = 2\pi \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \delta(\tau - i(\sigma + n)) \quad (\sigma > 0).$$

In general, for all functions $\Lambda(\rho + i\tau)$ ($\rho \in \mathbb{R}$), for which the delta function along the imaginary axis is defined, (3.36) gives

$$\langle F_M(\sigma + i\tau), \Lambda(\rho + i\tau)\rangle = 2\pi \sum_{n=-\infty}^{\infty} a_n \left\langle \delta(\tau - i(\sigma + n), \Lambda(\rho + i\tau)\right\rangle. \tag{3.39}$$

This inner product is well defined for all those functions for which this infinite series converges. By using the shifting property of the delta function, we get

$$\langle \delta(\tau - i(\sigma + n), \Lambda(\rho + i\tau)\rangle = \Lambda(\rho - \sigma - n).$$

Equation (3.39) can be rewritten as

$$\langle F_M(\sigma + i\tau), \Lambda(\rho + i\tau)\rangle = 2\pi \sum_{n=-\infty}^{\infty} a_n \Lambda(\rho - \sigma - n). \tag{3.40}$$

Consider the inner product of the distributional representation with the set of functions

$$\left\{ y^{-(\rho+i\tau)u} \right\}_{u\in\mathbb{C}} \quad (y > 0). \tag{3.41}$$

It gives

$$\begin{aligned} \left\langle F_M(\sigma + i\tau), y^{-(\rho+i\tau)u} \right\rangle &= 2\pi \sum_{n=-\infty}^{\infty} a_n y^{-(\rho-\sigma-n)u} \\ &= 2\pi y^{(\sigma-\rho)u} \sum_{n=-\infty}^{\infty} a_n y^{nu}. \end{aligned} \tag{3.42}$$

Replacing y^u by w in (3.42), one can obtain the following equation

$$\int_{-\infty}^{\infty} F_M(\sigma + i\tau) w^{-(\sigma+i\tau)} d\tau = 2\pi f(w) \quad (\sigma \in S_f). \tag{3.43}$$

Remark 15 *If the function f has a Taylor series representation,*

$$f(t) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)t^n}{n!},$$

then $F_M(\sigma + i\tau)$, if it exists, has the distributional representation

$$F_M(\sigma + i\tau) = 2\pi \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} \delta(\tau - i(\sigma + n)), \tag{3.44}$$

and we have

$$\langle F_M(\sigma + i\tau), \Lambda(\rho + i\tau)\rangle = 2\pi \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} \Lambda(\rho - \sigma - n). \tag{3.45}$$

Or, equivalently,

$$\int_{-\infty}^{\infty} F_M(\sigma+i\tau)\Lambda(\rho+i\tau)d\tau = 2\pi\sum_{n=0}^{\infty}\frac{f^{(n)}(0)}{n!}\Lambda(\rho-\sigma-n), \tag{3.46}$$

$$\frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} F_M(s)\Lambda(\rho-\sigma+s)ds = \sum_{n=0}^{\infty}\frac{f^{(n)}(0)}{n!}\Lambda(\rho-\sigma-n). \tag{3.47}$$

3.6 Some other applications of the distributional representation

1. **Ramanujan's master theorem**

The formula in (3.47) can be considered as a generalization of Ramanujan's master theorem. To see that, take $f(t) = e^{-xt}, x > 0$, which has the Mellin transform $F_M(s) = x^{-s}\Gamma(s)$, when $\Re(s) > 0$. By letting $\rho = \sigma$, (3.47) gives

$$\frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} x^{-s}\Gamma(s)\Lambda(-s)ds = \sum_{n=0}^{\infty}\frac{(-1)^n x^n}{n!}\Lambda(n) \quad (\sigma > 0), \tag{3.48}$$

which is Ramanujan's master theorem. The identity (3.48) is valid in the Hardy space of analytic functions [22] which is defined as follows. Take $0 < \delta < 1$ and $H(\delta) := \{s = \sigma + i\tau : \sigma \geq -\delta\}$ as half space. Suppose that $0 < A < \pi$ and

$$L(A, P, \delta) := \{\phi(s) : |\phi| \leq Ce^{P\sigma + A|\tau|}, s \in H(\delta)\}.$$

The space $L(A, P, \delta)$ is called the Hardy space of analytic functions.

Remark 16 *The Hardy space restricts the parameter A to lie in $(0, \pi)$. In [7], the Ramanujan master theorem has been extended to a wider class of functions.*

2. **Euler's reflection formula**

If we let $f(t) = \sin(xt)$, with $x > 0$, we have the Mellin transform

$$F_M(s) = x^{-s}\Gamma(s)\sin(\frac{\pi}{2}s) \quad (-1 < \Re(s) < 1).$$

By using the Taylor series

$$f(t) = \sin(xt) = \sum_{n=0}^{\infty} \frac{(-1)^n (x)^{2n+1} (t)^{2n+1}}{(2n+1)!},$$

from (3.47) with $\rho = \sigma$ and $\Lambda(s) = \frac{\Gamma(1-s)}{\Gamma(\frac{1-s}{2})}$, we have

$$\begin{aligned} \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} x^{-s}\Gamma(s)\sin(\frac{\pi}{2}s)\frac{\Gamma(1-s)}{\Gamma(\frac{1-s}{2})}ds &= \sum_{n=0}^{\infty} \frac{(-1)^n (x)^{2n+1}}{(2n+1)!\Gamma(\frac{2+2n}{2})}\Gamma(2n+2) \\ &= \sum_{n=0}^{\infty}(-1)^n \frac{(x)^{2n+1}}{n!} = xe^{-x^2}, \end{aligned}$$

Taking the Mellin transform for both sides, we get

$$\begin{aligned} \Gamma(s)\sin(\frac{\pi}{2}s)\frac{\Gamma(1-s)}{\Gamma(\frac{1-s}{2})} &= M\left[xe^{-x^2}; s\right] \\ &= \int_0^{\infty} x^s e^{-x^2} dx = \frac{1}{2}\int_0^{\infty} t^{\frac{s-1}{2}} e^{-t} dt \\ &= \frac{1}{2}\Gamma(\frac{s+1}{2}) \quad (\Re(s) > -1), \end{aligned}$$

which can be written as

$$\Gamma(s)\sin(\frac{\pi}{2}s)\Gamma(1-s) = \frac{1}{2}\Gamma(\frac{s+1}{2})\Gamma(\frac{1-s}{2}) \quad (\Re(s) > -1). \tag{3.49}$$

Replacing s by $2s$ in (3.49) and using the duplication formula

$$\Gamma(s)\Gamma\left(s + \frac{1}{2}\right) = 2^{1-2s}\sqrt{\pi}\,\Gamma(2s),$$

we get Euler's reflection formula

$$\Gamma(s)\Gamma(1-s) = \frac{\pi}{\sin(\pi s)}.$$

3. The Riemann functional equation

A large number of proofs of the functional equation for the Riemann zeta function, (1.45), are known. Indeed, in [46], several proofs are given. In [9], the authors have proved that the Riemann functional equation can be recovered by the Mellin transforms of essentially all the absolutely integrable functions. Here, we prove the Riemann functional equation by using the distributional representation.

If we take $f(t) = \sin(xt)$, with $x > 0$, and $\Lambda(s) = 2(2\pi)^{s-1}\Gamma(1-s)\zeta(1-s)$, then from (3.47), with $\rho = \sigma$, we have

$$\begin{aligned}
&\frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} x^{-s}\Gamma(s)\sin(\frac{\pi}{2}s)2(2\pi)^{s-1}\Gamma(1-s)\zeta(1-s)ds \\
= &\sum_{n=0}^{\infty}\frac{(-1)^n(x)^{2n+1}}{(2n+1)!}\Gamma(2n+2)2(\frac{1}{2\pi})^{2n+2}\zeta(2n+2) \\
= &\sum_{n=0}^{\infty}\frac{(-1)^n}{\pi}(\frac{x}{2\pi})^{2n+1}\zeta(2n+2) \\
= &\frac{1}{e^x-1} - \frac{1}{x} + \frac{1}{2}.
\end{aligned}$$

Take the Mellin transform for both sides, we have

$$\begin{aligned}
\Gamma(s)\sin(\frac{\pi}{2}s)2(2\pi)^{s-1}\Gamma(1-s)\zeta(1-s) &= \int_0^{\infty} x^{s-1}\left[\frac{1}{e^x-1} - \frac{1}{x} + \frac{1}{2}\right]dx \\
&= \Gamma(s)\zeta(s) \\
(-1 &< \Re(s) < 0).
\end{aligned}$$

By analytical continuation we get the Riemann functional equation

$$\zeta(s) = 2^s \pi^{s-1} \sin(\frac{\pi}{2}s)\Gamma(1-s)\zeta(1-s).$$

In conclusion, generally, distributions are taken as defined on the real domain only. Recently, they have been extended to the complex domain and applied to functions of complex variables that include the "special functions of mathematical physics" and those of number theory. The applications have provided new formulas for the properties of many of these functions.

There are several representations for generalized hypergeometric functions, ${}_pF_q$, available in the literature, for example, series and integral representations. In this chapter , we have represented them as a series of Dirac delta functions. This representation has led to some new integral formulas about generalized hypergeometric functions as well as for Gauss and confluent hypergeometric functions.

The distributional representation for any Mellin transformable function which has a Laurent or Taylor series has been obtained. An application of the distributional representation gave a formula which can be considered as a generalization of Ramanujan's master theorem. Some applications of the distributional representation are used to find Euler's reflection formula and the Riemann functional equation.

The distributional representation allows us to evaluate some integrals without using the residue theorem. It allows us to write the integral in terms of a series. It will lead to some new formulas and properties for special functions.

CHAPTER 4

APPLICATIONS OF PARSEVAL'S FORMULA FOR THE MELLIN TRANSFORM

In this chapter we apply the Parseval formula for the Mellin transform to generalized gamma, extended Beta, extended Gauss hypergeometric, and extended confluent hypergeometric functions.

Recall the Parseval formula for Mellin transform

$$\int_0^\infty f(x)g(x)dx = \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} F_M\left(s\right) G_M\left(1-s\right) ds, \tag{4.1}$$

and more generally

$$\int_0^\infty f(x)g(x)x^{p-1}dx = \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} F_M\left(s\right)G_M\left(p-s\right)ds. \tag{4.2}$$

By putting $s=\sigma+i\tau$ and $p=\sigma+\rho$, the Parseval formula (4.2) can be written as:

$$\int_{-\infty}^{\infty} F_M\left(\sigma+i\tau\right)G_M\left(\rho-i\tau\right)d\tau = 2\pi\int_0^\infty f(x)g(x)x^{\sigma+\rho-1}dx = 2\pi M[fg,\sigma+\rho]. \tag{4.3}$$

4.1 Applications of Parseval's formula to the gamma and generalized gamma functions

In this section we apply Parseval's formula in (4.1) for the Mellin transform and its generalization in (4.2) to the gamma and generalized gamma functions. We find some integral identities.

If we take $f(x)=x^a e^{-x}$ and $g(x)=y^b x^{b-1}e^{-xy}$, where $y>0$, we have the Mellin transforms $M[f,s]=\Gamma(s+a), \Re(a+s)>0$ and

$$M[g,s]=y^b\int_0^\infty x^{s+b-2}e^{-xy}dx = y^{1-s}\Gamma(s+b-1) \quad (\Re(s+b)>1). \tag{4.4}$$

Now, applying (4.1), we find that

$$\begin{aligned}\frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty}\Gamma(s+a)y^s\Gamma(b-s)ds &= y^b\int_0^\infty x^{a+b-1}\exp\left\{-x(1+y)\right\}dx\\ &= \frac{y^b\Gamma(a+b)}{(1+y)^{a+b}},\end{aligned} \tag{4.5}$$

provided that $-\Re(a)<\sigma<\Re(b)$. By taking $a=0$ and replacing y by y^{-1} in

(4.5) we find

$$\frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty}\Gamma(s)\Gamma(b-s)y^{-s}ds=\frac{\Gamma(b)}{(1+y)^b}\quad(0<\sigma<\Re(b)). \tag{4.6}$$

Taking $y=1$ in (4.5) we get

$$\frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty}\Gamma(s+a)\Gamma(b-s)ds=\frac{\Gamma(a+b)}{2^{a+b}}, \tag{4.7}$$

and with $a=b>0$, we obtain the well-known "norm squared" of Γ

$$\int_{-\infty}^{\infty}\left|\Gamma\left(a+i\tau\right)\right|^2 d\tau=\pi 2^{1-2a}\Gamma\left(2a\right). \tag{4.8}$$

If we take $f(x)=x^a e^{-x}$ and $g(x)=y^b x^{-b-1}e^{-\frac{y}{x}}$, where $y>0$, we have the Mellin transforms

$$M[g,s]=y^b\int_0^{\infty}x^{s-b-2}e^{-\frac{y}{x}}dx=y^{s-1}\Gamma(b+1-s)\quad(\Re(b+1-s)>0). \tag{4.9}$$

Making use of (4.1), we have

$$\begin{aligned}\frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty}\Gamma(s+a)\Gamma(s+b)y^{-s}ds &= y^b\int_0^{\infty}x^{a-b-1}\exp\left\{-x-\frac{y}{x}\right\}dx\\ &= y^b\Gamma_y(a-b),\end{aligned} \tag{4.10}$$

where $\Gamma_y(s)$ is the generalized gamma function defined by (1.19). When $a=b$ we find the formula

$$\frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty}\Gamma^2(s+a)y^{-s}ds=y^a\Gamma_y(0)\quad(\Re(a+\sigma)>0). \tag{4.11}$$

By using (1.20), (4.11) can be written in terms of the Macdonald function to give

$$\frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty}\Gamma^2(s+a)y^{-s}ds = 2y^aK_0(2\sqrt{y}) \quad (\Re(a+\sigma)>0). \tag{4.12}$$

Using Mellin transform Parseval's identity (4.3) for $f(t)=e^{-t-\frac{b}{t}}$ and $g(t)=e^{-t-\frac{y}{t}}$, we get

$$\begin{aligned}\int_{-\infty}^{\infty}&\Gamma_b\left(\sigma+i\tau\right)\Gamma_y\left(\rho-i\tau\right)d\tau = 2\pi M[fg,\sigma+\rho]\\ &= 2\pi\int_0^{\infty}t^{\sigma+\rho-1}e^{-2t-\frac{b+y}{t}}dt = \pi 2^{1-\sigma-\rho}\Gamma_{2(b+y)}\left(\sigma+\rho\right)\end{aligned}$$

$$(\Re(b)\geq 0, \Re(\sigma)>0 \text{ if } b=0) \text{ and } (\Re(y)\geq 0, \Re(\rho)>0 \text{ if } y=0). \tag{4.13}$$

Using the relation between the generalized gamma function and the Macdonald function (1.20), we can rewrite (4.13) as

$$\begin{aligned}\int_{-\infty}^{\infty}&(\frac{b}{y})^{\frac{i\tau}{2}}K_{\sigma+i\tau}(2\sqrt{b})K_{\rho-i\tau}(2\sqrt{y})d\tau\\ &= \pi 2^{-\frac{\sigma+\rho}{2}}(1+\frac{b}{y})^{\frac{\rho}{2}}(1+\frac{y}{b})^{\frac{\sigma}{2}}K_{\sigma+\rho}(2\sqrt{2(b+y)})\end{aligned}$$

$$(\Re(b)\geq 0, \Re(\sigma)>0 \text{ if } b=0) \text{ and } (\Re(y)\geq 0, \Re(\rho)>0 \text{ if } y=0). \tag{4.14}$$

The following results which were obtained earlier in [6] and [41] can be deduced as special cases of the above formulas:

$$\int_{-\infty}^{\infty}\Gamma_b\left(\sigma+i\tau\right)\Gamma_b\left(\rho-i\tau\right)d\tau = \pi 2^{1-\sigma-\rho}\Gamma_{4b}\left(\sigma+\rho\right)$$

$$(\Re(b)\geq 0, \sigma,\rho>0 \text{ if } b=0), \tag{4.15}$$

$$\int_{-\infty}^{\infty} \left|\Gamma_b\left(\sigma+i\tau\right)\right|^2 d\tau = \pi 2^{1-2\sigma}\Gamma_{4b}\left(2\sigma\right)$$
$$(\Re(b) \geq 0, \sigma > 0 \text{ if } b = 0), \tag{4.16}$$

$$\int_{-\infty}^{\infty} \Gamma\left(\sigma+i\tau\right)\Gamma_b\left(\rho-i\tau\right) d\tau = \frac{\pi}{2^{\sigma+\rho-1}}\Gamma_{2b}\left(\sigma+\rho\right)$$
$$(\Re(b) \geq 0, \sigma, \rho > 0 \text{ if } b = 0), \tag{4.17}$$

$$\int_{-\infty}^{\infty} K_{\sigma+i\tau}(2\sqrt{b})K_{\rho-i\tau}(2\sqrt{b})d\tau = \pi K_{\sigma+\rho}(4\sqrt{b})$$
$$(\Re(b) \geq 0, \sigma, \rho > 0 \text{ if } b = 0), \tag{4.18}$$

$$\int_{-\infty}^{\infty} \left|K_{\sigma+i\tau}(2\sqrt{b})\right|^2 d\tau = \pi K_{2\sigma}(4\sqrt{b})$$
$$(\Re(b) \geq 0, \sigma > 0 \text{ if } b = 0), \tag{4.19}$$

$$\int_{-\infty}^{\infty} \Gamma\left(\sigma+i\tau\right)\Gamma\left(\rho-i\tau\right) d\tau = \pi 2^{1-\sigma-\rho}\Gamma\left(\sigma+\rho\right) \quad (\sigma, \rho > 0) \tag{4.20}$$

and

$$\int_{-\infty}^{\infty} \left|\Gamma(\sigma+i\tau)\right|^2 d\tau = \pi 2^{1-2\sigma}\Gamma(2\sigma) \quad (\sigma > 0). \tag{4.21}$$

4.2 Applications of Parseval's formula to the extended beta functions

By using the Mellin transform representation of the generalized gamma function and the extended Beta function, several integrals of products involving the generalized gamma and the extended Beta functions are obtained.

By making the substitution $t = \sqrt{b}u$ in (1.19) we obtain

$$\begin{aligned} b^{\frac{-s}{2}}\Gamma_b(s) &= \int_0^\infty u^{s-1} e^{-\sqrt{b}\left(u-\frac{1}{u}\right)} du \\ &= M(e^{-\sqrt{b}\left(t-\frac{1}{t}\right)}; s) \quad (\Re(b) \geq 0, \Re(s) > 0 \text{ if } b = 0), \quad (4.22) \end{aligned}$$

where we define $\sqrt{b}$ by its principal value. By letting $u = \frac{t}{1-t}$ in (1.26), $B(s, \alpha - s; p)$ can be written as a Mellin transform

$$\begin{aligned} B(s, \alpha - s; p) &= e^{-2p} \int_0^\infty u^{s-1} \frac{\exp -p\left[u + \frac{1}{u}\right]}{(1+u)^\alpha} du \\ &= e^{-2p} M\left[\frac{\exp -p\left[t + \frac{1}{t}\right]}{(1+t)^\alpha}; s\right] \\ (\Re(p) &\geq 0, \Re(\alpha) > \Re(s) > 0 \text{ if } p = 0). \qquad (4.23) \end{aligned}$$

Using Parseval's formula (4.3) for (4.22) and (4.23) leads to

$$\begin{aligned} &\int_{-\infty}^\infty B(\sigma + i\tau, \alpha - \sigma - i\tau; p) b^{\frac{-(\rho - i\tau)}{2}} \Gamma_b(\rho - i\tau)\, d\tau \\ &= 2\pi e^{-2p} \int_0^\infty t^{\sigma+\rho-1} \frac{\exp\left[-p\left(t + \frac{1}{t}\right)\right] \exp\left[-\sqrt{b}\left(t + \frac{1}{t}\right)\right]}{(1+t)^\alpha} dt \\ &= 2\pi e^{2\sqrt{b}} B(\sigma + \rho, \alpha - \sigma - \rho; p + \sqrt{b}) \end{aligned}$$

$$(\Re(b) \geq 0, \rho > 0 \text{ if } b = 0) \text{ and } (\Re(p) \geq 0, \Re(\alpha) > \sigma > 0 \text{ if } p = 0). \qquad (4.24)$$

If we let $f(t) = \frac{e^{-2b}\exp\left[-b\left(t+\frac{1}{t}\right)\right]}{(1-t)^w}$ and $g(t) = \frac{e^{-2p}\exp\left[-p\left(t+\frac{1}{t}\right)\right]}{(1-t)^v}$, we have the Mellin transform

$$M[\frac{e^{-2b}\exp\left[-b\left(t + \frac{1}{t}\right)\right]}{(1-t)^w}, s] = B(s, w - s; b)$$

$$(\Re(b) \geq 0, \Re(w) > \Re(s) > 0 \text{ if } b = 0,) \qquad (4.25)$$

and

$$M[\frac{e^{-2p}\exp\left[-p\left(t+\frac{1}{t}\right)\right]}{(1-t)^{v}},s]=B(s,v-s;p)$$

$$(\Re(p)\geq 0,\Re(v)>\Re(s)>0 \text{ if } p=0).$$

From (4.3), we then find that

$$\begin{aligned}
&\int_{-\infty}^{\infty} B(\sigma+i\tau,w-\sigma-i\tau;b)B(\rho-i\tau,v-\rho+i\tau;p)d\tau \\
&= 2\pi e^{-2(b+p)}\int_{0}^{\infty} t^{\sigma+\rho-1}\frac{\exp\left[-(b+p)\left(t+\frac{1}{t}\right)\right]}{(1-t)^{w+v}}dt \\
&= 2\pi B(\sigma+\rho,w+v-\sigma-\rho;b+p)
\end{aligned}$$

$$(\Re(b)\geq 0,\Re(w)>\sigma>0 \text{ if } b=0) \text{ and } (\Re(p)\geq 0,\Re(v)>\rho>0 \text{ if } p=0), \tag{4.26}$$

where $B(s,w;p)$ is the extended Beta function defined by (1.26)

$$\begin{aligned}
B(s,w;p) &:= \int_{0}^{1} t^{s-1}(1-t)^{w-1}\exp\left[\frac{-p}{t(1-t)}\right]dt \\
(\Re(p) &\geq 0,\Re(s),\Re(w)>0 \text{ if } p=0).
\end{aligned}$$

Putting $p=b=0$ in (4.26), we get

$$\begin{aligned}
&\int_{-\infty}^{\infty} B(\sigma+i\tau,w-\sigma-i\tau)B(\rho+i\tau,v-\rho-i\tau)d\tau \\
&= 2\pi B(\sigma+\rho,w+v-\sigma-\rho) \quad (\Re(w)>\sigma>0,\Re(v)>\rho>0),
\end{aligned} \tag{4.27}$$

where $B(s,w)$ is the beta function defined by (1.14). By taking $w=2\sigma, v=2\rho$ in (4.27) we get the result obtained in [6]

$$\int_{-\infty}^{\infty}\left|\Gamma\left(\sigma+i\tau\right)\right|^{2}\left|\Gamma\left(\rho+i\tau\right)\right|^{2}d\tau = 2\pi\Gamma\left(2\sigma\right)\Gamma\left(2\rho\right)B(\sigma+\rho,\sigma+\rho). \tag{4.28}$$

4.3 Applications of Parseval's formula to the extended Gauss and confluent hypergeometric functions

In this section we apply Parseval's formula in (4.3) for Mellin transforms to the extended Gauss and confluent hypergeometric functions.

The extended Gauss hypergeometric function has the integral representation (see (1.39))

$$F_p(a,b;c;z) = \frac{e^{-2p}}{B(b,c-b)} \int_0^\infty \frac{t^{b-1}(1-t)^{a-c}}{[1+t(1-z)]^a} \exp\left[-p\left(t+\frac{1}{t}\right)\right] dt$$

$$(\Re(c) > \Re(b) > 0, \Re(p) \geq 0, |\arg(1-z)| < \pi \text{ if } p = 0),$$

which means

$$B(b,c-b)F_p(a,b;c;z) = M[\frac{e^{-2p}(1-t)^{a-c}\exp\left[-p\left(t+\frac{1}{t}\right)\right]}{[1+t(1-z)]^a}; b]$$

$$(\Re(c) > \Re(b) > 0, \Re(p) \geq 0, |\arg(1-z)| < \pi \text{ if } p = 0).$$

Using (4.3) we have

$$\begin{aligned}
&\int_{-\infty}^{\infty} B(\sigma+i\tau, c-\sigma-i\tau)F_p(a,\sigma+i\tau;c;z) \\
&\qquad B(\rho-i\tau, l-\rho+i\tau)F_q(d,\rho-i\tau;l;z)d\tau \\
&= 2\pi e^{-2(b+p)} \int_0^\infty t^{\sigma+\rho-1} \frac{(1-t)^{a+d-c-l}\exp\left[-(b+p)\left(t+\frac{1}{t}\right)\right]}{[1+t(1-z)]^{a+d}} dt \\
&= 2\pi B(\sigma+\rho, c+l-\sigma-\rho)F_{p+q}(a+d,\sigma+\rho;c+l;z)
\end{aligned}$$

$$(\Re(c) > \sigma > 0, \Re(p) \geq 0, |\arg(1-z)| < \pi \text{ if } p = 0),$$

$$(\Re(l) > \rho > 0, \Re(q) \geq 0, |\arg(1-z)| < \pi \text{ if } q = 0).$$

Similarly for the extended confluent hypergeometric function. It has an integral representation (see (1.41))

$$\phi_p(b;c;z) = \frac{\exp(-2p)}{B(b,c-b)} \int_0^\infty t^{b-1}(1+t)^{-c} \exp\left[z\left(\frac{t}{t+1}\right) - p\left(t+\frac{1}{t}\right)\right] dt$$
$$(\Re(p) \geq 0, \Re(c) > \Re(b) > 0 \text{ if } p = 0). \tag{4.29}$$

This means that

$$B(b,c-b)\phi_p(b;c;z) = \exp(-2p)M[(1+t)^{-c} \exp\left[z\left(\frac{t}{t+1}\right) - p\left(t+\frac{1}{t}\right)\right]; b].$$

Using (4.3) we have

$$\int_{-\infty}^{\infty} B(\sigma+i\tau, c-\sigma-i\tau)\phi_p(\sigma+i\tau;c;z)B(\rho-i\tau, l-\rho+i\tau)\phi_q(\rho-i\tau;l;w)d\tau$$
$$= 2\pi e^{-2(q+p)} \int_0^\infty t^{\sigma+\rho-1} \frac{(1-t)^{-c-l} \exp\left[(z+w)\left(\frac{t}{t+1}\right) - (q+p)\left(t+\frac{1}{t}\right)\right]}{[1+t(1-z)]^{a+d}} dt$$
$$= 2\pi B(\sigma+\rho, c+l-\sigma-\rho)\phi_{p+q}(\sigma+\rho;c+l;z+w)$$
$$(\Re(p) \geq 0, \Re(c) > \sigma > 0 \text{ if } p = 0) \text{ and } (\Re(q) \geq 0, \Re(l) > \rho > 0 \text{ if } q = 0).$$

4.4 Some applications of Parseval's formula for the incomplete Mellin transform

In this section we apply the Parseval formula for the incomplete Mellin transform. Suppose $f \in H(\sigma_2;\sigma_1)$, for a positive real number x, we define the lower incomplete Mellin transform $M_1[f;s;x] = F_{M_1}(s,x)$ by

$$M_1[f;s;x] = F_{M_1}(s,x) = \int_0^x t^{s-1} f(t)dt \qquad (\sigma_1 < \Re(s) < \infty), \tag{4.30}$$

and the upper incomplete Mellin transform $M_2[f;s] = F_{M_2}(s)$ by

$$M_2[f;s;x] = F_{M_2}(s,x) = \int_x^{\infty} t^{s-1} f(t)dt \qquad (-\infty < \Re(s) < \sigma_2). \tag{4.31}$$

It is clear that

$$F_{M_1}(s,\infty) = F_{M_2}(s,0) = F_M(s)$$

and

$$F_{M_1}(s,x) + F_{M_2}(s,x) = F_M(s).$$

Now, let $g(t)$ be the Heaviside step function, $H(y-t)$, for some positive real number y

$$g(t) := H(y-t) = \begin{cases} 1 & 0 \le t < y \\ 0 & t \ge y. \end{cases} \tag{4.32}$$

The Mellin transform of $g(t)$ is given by

$$M[g(t),s) = \int_0^{\infty} t^{s-1} H(y-t)dt = \int_0^{y} t^{s-1} dt = \frac{y^s}{s} \qquad (\Re(s) > 0, y > 0). \tag{4.33}$$

From (4.3), we find that

$$\begin{aligned} \int_{-\infty}^{\infty} F_M(\sigma + i\tau) \frac{y^{\rho - i\tau}}{\rho - i\tau} d\tau &= 2\pi \int_0^{\infty} t^{\sigma+\rho-1} f(t) H(y-t)dt \\ &= 2\pi \int_0^{y} t^{\sigma+\rho-1} f(t)dt \\ &= 2\pi F_{M_1}(\sigma+\rho, y). \end{aligned} \tag{4.34}$$

Similarly, if we let $r(t) = H(t-y)$ for some positive real number y

$$r(t) := H(t-y) = \begin{cases} 0 & 0 \le t < y \\ 1 & t \ge y. \end{cases}$$

The Mellin transform of $r(t)$ is given by

$$M[r(t), s] = \int_0^\infty t^{s-1} H(t-y) dt = \int_y^\infty t^{s-1} dt = -\frac{y^s}{s} \qquad (\Re(s) < 0).$$

Applying (4.3), we have

$$\begin{aligned} -\int_{-\infty}^{\infty} F_M(\sigma + i\tau) \frac{y^{\rho - i\tau}}{\rho - i\tau} d\tau &= 2\pi \int_0^\infty t^{\sigma+\rho-1} f(t) H(t-y) dt \\ &= 2\pi \int_y^\infty t^{\sigma+\rho-1} f(t) dt \\ &= 2\pi F_{M_2}(\sigma + \rho, y). \end{aligned}$$

Replacing ρ by $-\rho$ yields

$$\int_{-\infty}^{\infty} F_M(\sigma + i\tau) \frac{y^{-\rho-i\tau}}{\rho + i\tau} d\tau = 2\pi F_{M_2}(\sigma - \rho, y)$$
$$(\rho > 0, \sigma \in S_f, y > 0), \tag{4.35}$$

where S_f is the strip of analyticity of $M[f; s]$ as in the definition (1.80). Assume there exists a positive real number $0 < \rho \in S_f$ and take $\sigma = \rho$ in (4.35) to get

$$\int_{-\infty}^{\infty} \frac{F_M(\sigma + i\tau) y^{-i\tau}}{\sigma + i\tau} d\tau = 2\pi y^\sigma F_{M_2}(0, y) \qquad (0 < \sigma \in S_f). \tag{4.36}$$

Now, assume that there exists a positive real number $\rho > 0$ such that $\rho + 1 \in S_f$, then we have

$$\int_{-\infty}^{\infty} \frac{F_M(\sigma + 1 + i\tau) y^{-i\tau}}{\sigma + i\tau} d\tau = 2\pi y^\sigma F_{M_2}(1, y) \qquad (\sigma \in S_f). \tag{4.37}$$

Now, putting $f(t) = e^{-t}$, we find that $F_{M_1}(s, y) = \gamma(s, y)$, $\Re(s) > 0$, and $F_{M_2}(s, y) = \Gamma(s, y)$ and using (4.34) we get the relation between the complete and incomplete gamma functions

$$\int_{-\infty}^{\infty} \frac{\Gamma(\sigma+i\tau)\, y^{-i\tau}}{\rho-i\tau} d\tau = 2\pi y^{-\rho}\gamma(\sigma+\rho,y) \qquad (\sigma,\rho,y>0). \tag{4.38}$$

In particular for $y=1$, we have

$$\int_{-\infty}^{\infty} \frac{\Gamma(\sigma+i\tau)}{\rho-i\tau} d\tau = 2\pi\gamma(\sigma+\rho,1) \qquad (\sigma,\rho>0). \tag{4.39}$$

By using (4.35), we obtain similar results for $\Gamma(s,y)$

$$\int_{-\infty}^{\infty} \frac{\Gamma(\sigma+i\tau)\, y^{-i\tau}}{\rho+i\tau} d\tau = 2\pi y^{\rho}\Gamma(\sigma-\rho,y) \qquad (\sigma,\rho,y>0) \tag{4.40}$$

and

$$\int_{-\infty}^{\infty} \frac{\Gamma(\sigma+i\tau)}{\rho+i\tau} d\tau = 2\pi\Gamma(\sigma-\rho,1) \qquad (\sigma,\rho>0).$$

The following corollary can be considered as a generalization of the result obtained in [41, (3.3)].

Corollary 17 *For $\sigma,\rho,y>0$, the following identities hold:*

$$\begin{aligned} \int_{-\infty}^{\infty} \Gamma_b(\sigma+i\tau)\,\gamma(\rho-i\tau,y)d\tau &= 2^{1-(\sigma+\rho)}\pi\gamma_{2b}(\sigma+\rho,2y), & (4.41)\\ \int_{-\infty}^{\infty} \Gamma(\sigma+i\tau)\,\gamma(\rho-i\tau,y)d\tau &= 2^{1-(\sigma+\rho)}\pi\gamma(\sigma+\rho,2y). & (4.42) \end{aligned}$$

Proof. By taking $f(t)=e^{-t-\frac{b}{t}}$ and $g(t)=H(y-t)e^{-t}$ and using (4.3) we find (4.41). Putting $b=0$ in (4.41) we get (4.42). ∎

Theorem 18 *Let f be a Mellin transformable function with the strip of analyticity S_f . Let $(y_k)_{k=1}^{\infty}$ be a sequence of real positive numbers. Let $(a_k)_{k=1}^{\infty}$ be a sequence of complex numbers. Assume that the series $L(s)=\sum_{k=1}^{\infty}\frac{a_k}{(y_k)^s}$*

converges absolutely for $\Re(s) > \rho_0$*. Then*

$$\int_{-\infty}^{\infty} F_M(\sigma + i\tau)\, L(\rho + i\tau) d\tau = 2\pi \sum_{k=1}^{\infty} a_k y_k^{\sigma-\rho} f(y_k) \quad (\sigma \in S_f, \rho > \rho_0). \tag{4.43}$$

Proof. Define a sequence of distributions $g_k(t)$ by

$$g_k(t) = a_k \delta(t - y_k).$$

Using the definition of the Dirac delta function (1.102), we have the Mellin transform

$$M[g_k(t), s] = < a_k \delta(t - y_k), t^{s-1} > = a_k y_k^{s-1} \qquad (-\infty < \Re(s) < \infty).$$

Applying (4.3) we have

$$\begin{aligned} \int_{-\infty}^{\infty} F_M(\sigma + i\tau)\, a_k y_k^{\rho - i\tau - 1} d\tau &= 2\pi M[f g_k, \sigma + \rho] \\ &= 2\pi a_k \int_0^{\infty} t^{\sigma+\rho-1} \delta(t - y_k) f(t) dt \\ &= 2\pi a_k y_k^{\sigma+\rho-1} f(y_k). \end{aligned}$$

Replace $\rho - 1$ by $-\rho$ and take the summation over k, we have

$$\sum_{k=1}^{\infty} \int_{-\infty}^{\infty} F_M(\sigma + i\tau)\, a_k y_k^{-\rho - i\tau} d\tau = 2\pi \sum_{k=1}^{\infty} a_k y_k^{\sigma-\rho} f(y_k). \tag{4.44}$$

Since $L(s)$ converges absolutely for $\Re(s) > \rho_0$, we have

$$\sum_{k=1}^{\infty} \int_{-\infty}^{\infty} \left| F_M(\sigma + i\tau)\, a_k y_k^{-\rho - i\tau} \right| d\tau \leqslant \sum_{k=1}^{\infty} \left| a_k y_k^{-\rho} \right| \int_{-\infty}^{\infty} \left| F_M(\sigma + i\tau) \right| d\tau < \infty \tag{4.45}$$

for $\sigma \in S_f, \rho > \rho_0$. Hence from (4.44)

$$\begin{aligned} 2\pi \sum_{k=1}^{\infty} a_k y_k^{\sigma-\rho} f(y_k) &= \sum_{k=1}^{\infty} \int_{-\infty}^{\infty} F_M(\sigma+i\tau)\, a_k y_k^{-\rho-i\tau} d\tau \\ &= \int_{-\infty}^{\infty} F_M(\sigma+i\tau) \sum_{k=1}^{\infty} a_k y_k^{-\rho-i\tau} d\tau \\ &= \int_{-\infty}^{\infty} F_M(\sigma+i\tau)\, L(\rho+i\tau) d\tau. \end{aligned}$$

The inversion of the order of summation and integration is justified by (4.45). So we obtain (4.43). ■

Corollary 19 *The following relations hold*

$$\int_{-\infty}^{\infty} \Gamma_b(\sigma+i\tau)\, \zeta(\rho+i\tau) y^{-(\sigma+i\tau)} d\tau = 2\pi \sum_{n=1}^{\infty} n^{\sigma-\rho} e^{-yn-\frac{b}{ny}}$$

$$(y > 0, \rho > 1, \Re(b) \geq 0, \sigma > 0 \text{ if } b = 0), \tag{4.46}$$

$$\int_{-\infty}^{\infty} \Gamma_b(\sigma+i\tau)\, \zeta(\rho+i\tau) d\tau = 2\pi \sum_{n=1}^{\infty} n^{\sigma-\rho} e^{-n-\frac{b}{n}}$$

$$(\rho > 1, \Re(b) \geq 0, \sigma > 0 \text{ if } b = 0), \tag{4.47}$$

$$\int_{-\infty}^{\infty} b^{\frac{\sigma+i\tau}{2}} K_{\sigma+i\tau}(2\sqrt{b}) \zeta(\rho+i\tau) d\tau = \pi \sum_{n=1}^{\infty} n^{\sigma-\rho} e^{-n-\frac{b}{n}}$$

$$(\Re(b) \geq 0, \rho > 1), \tag{4.48}$$

and

$$\int_{-\infty}^{\infty} K_{\sigma+i\tau}(2) \zeta(\sigma+i\tau) d\tau = \pi \sum_{n=1}^{\infty} e^{-n-\frac{1}{n}} \quad (\sigma > 1). \tag{4.49}$$

Proof. If we let $L(s) = \zeta(s)$ and $f(t) = e^{-yt-\frac{b}{t}}$, for $y, b > 0$, then we have the

Mellin transform

$$M[e^{-yt-\frac{b}{t}}, s] = \int_0^{\infty} t^{s-1} e^{-yt-\frac{b}{t}} dt = y^{-s} \Gamma_{by}(s) .$$

Applying (4.43), we get (4.46). To prove (4.47), let $y = 1$. Using the relation between the generalized gamma function and the Macdonald function (1.20) we prove (4.48). Put $b = 1$ and $\sigma = \rho$ in (4.48) to obtain (4.49). ■

Letting $a_k = 1$, $y_k = k$ in (4.43), we have

$$\int_{-\infty}^{\infty} F_M(\sigma + i\tau) \zeta(\rho + i\tau) d\tau = 2\pi \sum_{n=1}^{\infty} \frac{f(n)}{n^{\rho-\sigma}} \quad (\sigma \in S_f, \rho > 1). \tag{4.50}$$

We also get the classical result

$$\sum_{n=1}^{\infty} f(n) = \frac{1}{2\pi i} \int_{\sigma-i\infty}^{\sigma+i\infty} F_M(s) \zeta(s) ds, \tag{4.51}$$

by letting $\sigma = \rho$ and making the substitution $s = \sigma + i\tau$.

Taking $f(t) = e^{-yt}$ for $y > 0$, we have the Mellin transform

$$F_M(s) = y^{-s}\Gamma(s) \qquad (\Re(s) > 0).$$

From (4.50), we have

$$\int_{-\infty}^{\infty} y^{-i\tau} \Gamma(\sigma + i\tau) \zeta(\rho + i\tau) d\tau = 2\pi y^{\sigma} \sum_{n=1}^{\infty} \frac{e^{-yn}}{n^{\rho-\sigma}} \qquad (\sigma > 0, \rho > 1, y > 0) \tag{4.52}$$

and the special case $\sigma = \rho$ leads to

$$\int_{-\infty}^{\infty} y^{-i\tau} \Gamma(\sigma + i\tau) \zeta(\sigma + i\tau) d\tau = 2\pi y^{\sigma} \sum_{n=1}^{\infty} e^{-yn} = \frac{2\pi y^{\sigma}}{e^{y} - 1} \qquad (\sigma > 1, y > 0). \tag{4.53}$$

Putting $y = 1$ in (4.52), we get

$$\int_{-\infty}^{\infty} \Gamma\left(\sigma + i\tau\right)\zeta(\rho + i\tau)d\tau = 2\pi \sum_{n=1}^{\infty} \frac{e^{-n}}{n^{\rho-\sigma}} \qquad (\sigma > 0, \rho > 1). \tag{4.54}$$

Now, taking $f(t) = \left(e^t - 1\right)^{-1}$, we have the Mellin transform

$$F_M(s) = \Gamma(s)\zeta(s) \qquad (\Re(s) > 1).$$

From (4.50), we then have

$$\int_{-\infty}^{\infty} \Gamma\left(\sigma + i\tau\right)\zeta\left(\sigma + i\tau\right)\zeta(\rho + i\tau)d\tau = 2\pi \sum_{n=1}^{\infty} \frac{n^{\sigma-\rho}}{e^n - 1} \qquad (\sigma, \rho > 1). \tag{4.55}$$

We conclude this section by giving an application of (4.55).

Corollary 20 *The following relations hold*

$$\sum_{k=1}^{\infty} \frac{k^{2n+1}}{e^k - 1} = (2n+1)!\zeta(2n+2) - \frac{1}{2}\zeta(-2n-1) \qquad (n = 1, 2, 3, \ldots),$$

and

$$\sum_{k=1}^{\infty} \frac{k}{e^k - 1} = \frac{\pi^2}{6} - \frac{11}{24}. \tag{4.56}$$

Proof. Let $s = \sigma + i\tau$ in (4.55) and chose ρ and σ such that $\sigma - \rho = 2n + 1$. Evaluating the left side of (4.55) by using the residue theorem as follows:

$$\begin{aligned} L.H.S &= -i\int_{\sigma-i\infty}^{\sigma+i\infty} \Gamma\left(s\right)\zeta\left(s\right)\zeta(\rho - \sigma + s)ds \\ &= -i2\pi i \sum \text{Res}(\Gamma\left(s\right)\zeta\left(s\right)\zeta(-2n - 1 + s)) \\ &= 2\pi\{\Gamma\left(2n+2\right)\zeta\left(2n+2\right) + \Gamma\left(1\right)\zeta\left(-2n\right) \\ &\quad + \sum_{k=0}^{\infty} \frac{(-1)^k}{k!}\zeta(-k)\zeta(-2n-1-k)\} \\ &= 2\pi\left\{\Gamma\left(2n+2\right)\zeta\left(2n+2\right) + \zeta(0)\zeta(-2n-1)\right\}, \end{aligned}$$

where we used the fact that $\zeta\,(-2n) = 0$, for $n > 0$ and $\zeta(-k)\zeta(-2n-1-k) = 0$, for all integers $k \neq 0$. So, we obtain

$$\sum_{k=1}^{\infty} \frac{k^{2n+1}}{e^k - 1} = (2n+1)!\zeta(2n+2) - \frac{1}{2}\zeta(-2n-1) \qquad (n = 1, 2, 3, \ldots).$$

For $n = 0$, using the residue theorem again, we obtain

$$\begin{aligned} \sum_{k=1}^{\infty} \frac{k}{e^k - 1} &= \zeta(2) - \frac{1}{2}\zeta(-1) - \frac{1}{2} \\ &= \frac{\pi^2}{6} - \frac{11}{24}. \end{aligned}$$

∎

In recent years, several extensions of well known special functions have been considered by several authors. The generalized gamma, the extended beta, the extended Gauss hypergeometric and the extended confluent hypergeometric functions have been defined and proved to be useful in several applications. In this chapter, we have applied Parseval's identity for the Mellin transform to these functions. Several integrals of products involving these extended functions have been obtained. Some applications of Parseval's formula for the incomplete Mellin transform have been discussed.

CHAPTER 5

GENERALIZED EXTENDED FERMI-DIRAC AND BOSE-EINSTEIN FUNCTIONS

Generalization of special functions may prove more useful than the original special functions themselves. It is required that the results obtained from the generalization should be no less elegant than those from the original function. Criteria for generalizations of special functions are discussed in [35]. In this chapter, we generalize the eFD and eBE functions by inserting the regularizer $e^{-b/t}$ in the integral representations of the eFD and eBE functions (1.72) and

(1.73). The *generalized extended Fermi-Dirac* function can be thought of as a power series in which the coefficients are extended zeta functions divided by $n!$.

5.1 Generalized extended Fermi-Dirac function

In this section, we introduce and investigate the *generalized extended Fermi-Dirac* (geFD) function. We consider the following function

$$\vartheta(t;\nu;b) = \frac{e^{-\nu t}e^{\frac{-b}{t}}}{e^t+1} \qquad (\Re(\nu) > -1, \Re(b) \geq 0, t \geq 0), \tag{5.1}$$

$\vartheta(t;\nu;b)$ is integrable on every finite closed interval $[0,T]$ $(0 < T < \infty)$ and $\vartheta(t;\nu;b)$ belongs to $H(\infty,0)$. Therefore, the Weyl transform of $\vartheta(t;\nu;b)$, given by

$$\begin{aligned}\Theta_\nu(s;x;b) := W^s_{x^+}[\vartheta(t;\nu;b)] &= \frac{1}{\Gamma(s)}\int_0^\infty t^{s-1}\vartheta(t+x;\nu;b)dt\\ &= \frac{1}{\Gamma(s)}\int_x^\infty (t-x)^{s-1}\vartheta(t;\nu;b)dt\\ &= \frac{1}{\Gamma(s)}\int_0^\infty \frac{t^{s-1}e^{-\nu(t+x)}e^{\frac{-b}{t+x}}}{e^{t+x}+1}dt\\ &(x \geq 0, \Re(\nu) > -1, \Re(b) \geq 0, \Re(s) > 0 \text{ if } b = 0),\end{aligned} \tag{5.2}$$

is well defined. We call $\Theta_\nu(s;x;b)$ the generalized extended Fermi-Dirac function (geFD). It is interesting to note that for $b = 0$ the function is related to eFD by

$$\Theta_\nu(s;x;0) = \Theta_\nu(s;x). \tag{5.3}$$

Also, we have

$$\Theta_0(s;x;0) = \mathcal{F}_{s-1}(-x) \qquad (\Re(s) > 0, x \geq 0). \tag{5.4}$$

By putting $x = 0$ in (5.2), we have

$$\Theta_\nu(s;0;b) = \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} \vartheta(t;\nu;b) dt = \frac{1}{\Gamma(s)} \int_0^\infty \frac{t^{s-1} e^{-\nu t} e^{\frac{-b}{t}}}{e^t + 1} dt$$

$$(\Re(\nu) > -1, \Re(b) \geq 0, \Re(s) > 0 \text{ if } b = 0), \tag{5.5}$$

which can be written as

$$\Theta_\nu(s;0;b) = \frac{1}{\Gamma(s)} \int_0^\infty \frac{t^{s-1} e^{-(\nu+1)t} e^{\frac{-b}{t}}}{1 + e^{-t}} dt = (1 - 2^{1-s}) \zeta_b^*(s, \nu + 1)$$

$$(-1 < (\Re(\nu) \leq 0, \Re(b) \geq 0, \Re(s) > 0 \text{ if } b = 0), \tag{5.6}$$

where $\zeta_b^*(s,\nu)$ is the extended Hurwitz zeta function defined by (1.69).

Since the second integral in (5.5) remains absolutely convergent, we replace the exponential function $e^{-\nu t}$ by its series representation and interchange the order of integration and summation. We find from (5.5) that

$$\begin{aligned} \Theta_\nu(s;0;b) &= \sum_{n=0}^\infty \frac{1}{\Gamma(s)} \frac{(-\nu)^n}{n!} \int_0^\infty \frac{t^{s+n-1} e^{\frac{-b}{t}}}{e^t + 1} dt \\ &= \sum_{n=0}^\infty \frac{(-\nu)^n}{n!} (s)_n \left(\frac{1}{\Gamma(s+n)} \int_0^\infty \frac{t^{s+n-1} e^{\frac{-b}{t}}}{e^t + 1} dt \right), \end{aligned} \tag{5.7}$$

where $(s)_n$ is the Pochhammer symbol. However, the integral in (5.7) can be simplified in terms of extended Riemann zeta function (1.67) to give

$$\frac{1}{\Gamma(s+n)} \int_0^\infty \frac{t^{s+n-1} e^{\frac{-b}{t}}}{e^t + 1} dt = (1 - 2^{1-s-n}) \zeta_b^*(s+n)$$

$$(\Re(b) \geq 0, \Re(s) > 0 \text{ if } b = 0, n = 0, 1, 2, \ldots). \tag{5.8}$$

From (5.7) and (5.8), we find

$$\Theta_\nu(s;0;b)=\sum_{n=0}^{\infty}\frac{(-\nu)^n}{n!}(s)_n(1-2^{1-s-n})\zeta_b^*(s+n)$$

$$(\Re(\nu)>-1,\Re(b)\geq 0,\Re(s)>0 \text{ if } b=0). \tag{5.9}$$

Corollary 21 *The following relation is true*

$$\zeta_b^*(s,\nu+1)=\frac{1}{(1-2^{1-s})}\sum_{n=0}^{\infty}\frac{(-\nu)^n}{n!}(s)_n(1-2^{1-s-n})\zeta_b^*(s+n,1)$$

$$(-1<(\Re(\nu)\leq 0,\Re(b)\geq 0,\Re(s)>0 \text{ } if \text{ } b=0). \tag{5.10}$$

Proof. This follows directly by comparing (5.9) and (5.6). ■

Corollary 22 *We have*

$$\Theta_\nu(s;0;0)=\sum_{n=0}^{\infty}\frac{(-\nu)^n}{n!}(s)_n(1-2^{1-s-n})\zeta(s+n)$$

$$=(1-2^{1-s})\zeta_0^*(s,\nu+1) \qquad (\Re(s)>0,-1<(\Re(\nu)\leq 0). \tag{5.11}$$

Proof. By putting $b=0$ in (5.10) we get (5.11). ■

Corollary 23 *The following relation is valid*

$$\Theta_\nu(s;x;b)=\sum_{n=0}^{\infty}\frac{(-x)^n}{n!}(1-2^{1-s+n})\zeta_b^*(s-n,\nu+1)$$

$$(-1<\Re(\nu)\leq 0,x\geq 0,\Re(b)\geq 0,\Re(s)>0 \text{ } if \text{ } b=0).$$

Proof. Since $\Theta_\nu(s;x;b)$ is a Weyl transform of $\frac{e^{-\nu t}e^{\frac{-b}{t}}}{e^t+1}$, then

$$\Theta_\nu(s;x;b)=\sum_{n=0}^{\infty}\frac{(-x)^n}{n!}\Theta_\nu(s-n;0;b).$$

By using (5.6) we get

$$\Theta_\nu(s;x;b) = \sum_{n=0}^{\infty} \frac{(-x)^n}{n!}(1-2^{1-s+n})\zeta_b^*(s-n,\nu+1)$$
$$(-1 < \Re(\nu) \le 0, x \ge 0, \Re(b) \ge 0, \Re(s) > 0 \text{ if } b = 0).$$

■

By putting $\nu = 0$ we obtain

Corollary 24 *The following relation is valid*

$$\Theta_0(s;x;b) = \sum_{n=0}^{\infty} \frac{(-x)^n}{n!}(1-2^{1-s+n})\zeta_b^*(s-n)$$
$$(x \ge 0, \Re(b) \ge 0, \Re(s) > 0 \text{ if } b = 0). \tag{5.12}$$

Theorem 25 *The generalized extended Fermi-Dirac function $\Theta_\nu(s;x;b)$ can be expressed as an integral of itself as*

$$\begin{aligned}\Theta_\nu(s+\beta;x;b) &= \frac{1}{\Gamma(s)}\int_0^\infty t^{s-1}\Theta_\nu(\beta;x+t;b)dt \\ &= \frac{1}{\Gamma(\beta)}\int_0^\infty t^{\beta-1}\Theta_\nu(s;x+t;b)dt\end{aligned}$$
$$(x \ge 0, \Re(\nu) > -1, \Re(b) \ge 0, \min\{\Re(s), \Re(\beta)\} > 0 \text{ if } b = 0). \tag{5.13}$$

Proof. The proof follows directly from the semigroup property (1.99). ■

Corollary 26 *The extended Fermi-Dirac function $\Theta_\nu(s;x)$ can be expressed as* *an integral of itself as*

$$\Theta_\nu(s+\beta;x) = \frac{1}{\Gamma(s)}\int_0^\infty t^{s-1}\Theta_\nu(\beta;x+t)dt$$
$$= \frac{1}{\Gamma(\beta)}\int_0^\infty t^{\beta-1}\Theta_\nu(s;x+t)dt$$
$$(x \geq 0, \Re(\nu) > -1, \min\{\Re(s),\Re(\beta)\} > 0). \tag{5.14}$$

Proof. By putting $b = 0$ in (5.13), we get (5.14). ■

Corollary 27 *The Fermi-Dirac function $\mathcal{F}_{s-1}(x)$ can be written as an integral of itself as*

$$\mathcal{F}_{s+\beta-1}(x) = \frac{1}{\Gamma(s)}\int_0^\infty t^{s-1}\mathcal{F}_{\beta-1}(x-t)dt$$
$$= \frac{1}{\Gamma(\beta)}\int_0^\infty t^{\beta-1}\mathcal{F}_{s-1}(x-t)dt$$
$$(\min\{\Re(s),\Re(\beta)\} > 0, x \leq 0). \tag{5.15}$$

Proof. By putting $\nu = 0$ in (5.14) and using (1.74) we obtain

$$\mathcal{F}_{s+\beta-1}(-x) = \frac{1}{\Gamma(s)}\int_0^\infty t^{s-1}\mathcal{F}_{\beta-1}(-x-t)dt$$
$$= \frac{1}{\Gamma(\beta)}\int_0^\infty t^{\beta-1}\mathcal{F}_{s-1}(-x-t)dt$$
$$(\min\{\Re(s),\Re(\beta)\} > 0, x \geq 0).$$

We obtain (5.15) by replacing x by $-x$. ■

The operator representation in (5.15) provides a useful relation between the $\mathcal{F}_{s-1}(x)$ and its transformation.

5.2 Generalized extended Bose-Einstein function

In this section, we introduce and investigate the generalized extended Bose-Einstein function (geBE) $\Psi_\nu(s;x;b)$. This is a generalization of the extended Bose-Einstein functions $\Psi_\nu(s;,x)$ [39]. We consider a function $\psi(t,\nu,b)$ given by

$$\psi(t;\nu;b) = \frac{e^{-\nu t}e^{\frac{-b}{t}}}{e^t - 1} \qquad (\Re(\nu) > -1, \Re(b) \geq 0, t \geq 0), \tag{5.16}$$

and use this function to define $\Psi_\nu(s;x;b)$ as the Weyl transform

$$\begin{aligned}\Psi_\nu(s;x;b) := W^s_{x^+}[\psi(t;\nu;b)] &= \frac{1}{\Gamma(s)}\int_0^\infty t^{s-1}\psi(t+x;\nu;b)dt \\ &= \frac{1}{\Gamma(s)}\int_x^\infty (t-x)^{s-1}\psi(t;\nu;b)dt \\ &= \frac{1}{\Gamma(s)}\int_0^\infty \frac{t^{s-1}e^{-\nu(t+x)}e^{\frac{-b}{t+x}}}{e^{t+x}-1}dt \\ &(x \geq 0, \Re(\nu) > -1, \Re(b) \geq 0, \Re(s) > 1 \text{ if } b = 0).\end{aligned} \tag{5.17}$$

It is clear that $\Psi_\nu(s;x;0) = \Psi_\nu(s;x)$, and

$$\Psi_0(s;x;0) = B_{s-1}(-x) \qquad (\Re(s) > 1, x \geq 0). \tag{5.18}$$

By putting $x = 0$ in (5.17), we have

$$\begin{aligned}\Psi_\nu(s;0;b) &= \frac{1}{\Gamma(s)}\int_0^\infty t^{s-1}\psi(t;\nu;b)dt = \frac{1}{\Gamma(s)}\int_0^\infty \frac{t^{s-1}e^{-\nu t}e^{\frac{-b}{t}}}{e^t-1}dt \\ &= \frac{1}{\Gamma(s)}\int_0^\infty \frac{t^{s-1}e^{-(\nu+1)t}e^{\frac{-b}{t}}}{1-e^{-t}}dt = \zeta_b(s,\nu+1) \\ &(-1 < \Re(\nu) \leq 0, \Re(b) \geq 0, \Re(s) > 1 \text{ if } b = 0),\end{aligned} \tag{5.19}$$

where $\zeta_b(s,\nu)$ is the extended Hurwitz zeta function defined by

$$\zeta_b(s,\nu) = \frac{1}{\Gamma(s)} \int_0^\infty \frac{t^{s-1}e^{-\nu t}e^{\frac{-b}{t}}}{1-e^{-t}} dt$$
$$(0 < \Re(\nu) \leq 1, \Re(b) \geq 0, \Re(s) > 1 \text{ if } b = 0).$$

Corollary 28 *We have*

$$\Psi_\nu(s;x;b) = \sum_{n=0}^{\infty} \frac{(-x)^n}{n!} \zeta_b(s-n,\nu+1)$$
$$(x \geq 0, -1 < \Re(\nu) \leq 0, \Re(b) \geq 0, \Re(s) > 1 \textit{ if } b = 0). \tag{5.20}$$

Proof. Since $\Psi_\nu(s;x;b)$ is a Weyl transform of $\psi(t;\nu;b)$, then

$$\begin{aligned}\Psi_\nu(s;x;b) &= \sum_{n=0}^{\infty} \frac{(-x)^n}{n!} \Psi_\nu(s-n;0;b) \\ &= \sum_{n=0}^{\infty} \frac{(-x)^n}{n!} \zeta_b(s-n,\nu+1)\end{aligned}$$
$$(x \geq 0, -1 < \Re(\nu) \leq 0, \Re(b) \geq 0, \Re(s) > 1 \text{ if } b = 0).$$

■

By putting $\nu = 0$ in (5.20) we obtain

Corollary 29 *The following relation is valid*

$$\Psi_0(s;x;b) = \sum_{n=0}^{\infty} \frac{(-x)^n}{n!} \zeta_b(s-n)$$
$$(x \geq 0, \Re(b) \geq 0, \Re(s) > 1 \textit{ if } b = 0). \tag{5.21}$$

Since the integral in (5.19) is absolutely convergent, we replace the exponential function $e^{-\nu t}$ by its series representation and interchange the order of integration and summation. We find from (5.19) that

$$\begin{aligned}
\Psi_\nu(s;0;b) &= \frac{1}{\Gamma(s)}\int_0^\infty \frac{t^{s-1}e^{-\nu t}e^{\frac{-b}{t}}}{e^t-1}dt \\
&= \frac{1}{\Gamma(s)}\int_0^\infty \frac{t^{s-1}e^{\frac{-b}{t}}}{e^t-1}\sum_{n=0}^{\infty}\frac{(-\nu t)^n}{n!}dt \\
&= \sum_{n=0}^{\infty}\frac{1}{\Gamma(s)}\frac{(-\nu)^n}{n!}\int_0^\infty \frac{t^{s+n-1}e^{\frac{-b}{t}}}{e^t-1}dt \\
&= \sum_{n=0}^{\infty}\frac{(-\nu)^n}{n!}(s)_n\zeta_b(s+n)
\end{aligned}$$

$$(-1 < \Re(\nu) \leq 0, \Re(b) \geq 0, \Re(s) > 1 \text{ if } b = 0). \tag{5.22}$$

Corollary 30 *The following relations are valid*

$$\zeta_b(s,\nu+1) = \sum_{n=0}^{\infty}\frac{(-\nu)^n}{n!}(s)_n\zeta_b(s+n)$$

$$(-1 < \Re(\nu) \leq 0, \Re(b) \geq 0, \Re(s) > 1 \textit{ if } b = 0).$$

$$\Psi_\nu(s;0;0) = \zeta(s,\nu+1) = \sum_{n=0}^{\infty}\frac{(-\nu)^n}{n!}(s)_n\zeta(s+n)$$

$$(\Re(s) > 1, -1 < \Re(\nu) \leq 0).$$

5.3 Connection between the geFD and geBE functions

In this section we give the connection between the geFD and geBE functions. We deduce the connections between the eFD and eBE functions and the connections between the FD and BE functions as special cases.

Theorem 31 *The geFD and geBE functions are related by*

$$2^{1-s}\Psi_{\nu}(s,2x,2b) = \Psi_{2\nu}(s,x,b) - \Theta_{2\nu}(s,x,b)$$

$$(\Re(\nu) > -1, x \geq 0, \Re(b) \geq 0, \Re(s) > 1 \textit{ if } b = 0). \tag{5.23}$$

Proof. Since

$$\frac{e^{-2\nu t}e^{\frac{-b}{t}}}{e^{2t}-1} = \frac{1}{2}\left[\frac{e^{-2\nu t}e^{\frac{-b}{t}}}{e^{t}-1} - \frac{e^{-2\nu t}e^{\frac{-b}{t}}}{e^{t}+1}\right],$$

by taking the Weyl transform of both sides and using

$$W^{s}_{x^{+}}[f(2t)] = 2^{-s}W^{s}_{2x^{+}}[f(t)]$$

we obtain

$$2^{1-s}\Psi_{\nu}(s,2x,2b) = \Psi_{2\nu}(s,x,b) - \Theta_{2\nu}(s,x,b)$$

$$(\Re(\nu) > -1, x \geq 0, \Re(b) \geq 0, \Re(s) > 1 \text{ if } b = 0).$$

■

Theorem 32 *The geFD and geBE functions are related as follows:*

$$2^{s}\Theta_{\nu+1}(s,x,\frac{b}{2}) = \Psi_{\frac{\nu}{2}}(s,2x,b) - \Psi_{\frac{\nu+1}{2}}(s,2x,b).$$

$$(\Re(\nu) > -1, x \geq 0, \Re(b) \geq 0, \Re(s) > 1 \textit{ if } b = 0). \tag{5.24}$$

Proof. Replacing $\Psi_{\frac{\nu}{2}}(s,2x,b)$ and $\Psi_{\frac{\nu+1}{2}}(s,2x,b)$ in the right hand side by their integral representations we get

$$\Psi_{\frac{\nu}{2}}(s,2x,b)-\Psi_{\frac{\nu+1}{2}}(s,2x,b)$$
$$=\frac{1}{\Gamma(s)}\int_0^\infty \frac{t^{s-1}e^{-\frac{\nu}{2}(t+2x)}e^{\frac{-b}{t+2x}}}{e^{t+2x}-1}-\frac{t^{s-1}e^{-\frac{\nu+1}{2}(t+2x)}e^{\frac{-b}{t+2x}}}{e^{t+2x}-1}dt$$
$$=\frac{1}{\Gamma(s)}\int_{2x}^\infty (t-2x)^{s-1}e^{\frac{-b}{t}}\left(\frac{e^{-\frac{v}{2}t}-e^{-\frac{v+1}{2}t}}{e^t-1}\right)dt$$
$$=\frac{1}{\Gamma(s)}\int_{2x}^\infty (t-2x)^{s-1}e^{\frac{-b}{t}}e^{-\frac{v+1}{2}t}\left(\frac{e^{\frac{t}{2}}-1}{\left(e^{\frac{t}{2}}-1\right)\left(e^{\frac{t}{2}}+1\right)}\right)dt$$
$$=\frac{1}{\Gamma(s)}\int_{2x}^\infty \frac{(t-2x)^{s-1}e^{\frac{-b}{t}}e^{-\frac{v+1}{2}t}}{e^{\frac{t}{2}}+1}dt$$
$$(\Re(\nu)>-1, x\geq 0, \Re(b)\geq 0, \Re(s)>1 \text{ if } b=0).$$

The transformation $t=2\tau$ yields

$$\Psi_{\frac{\nu}{2}}(s,2x,b)-\Psi_{\frac{\nu+1}{2}}(s,2x,b)=\frac{2^s}{\Gamma(s)}\int_x^\infty \frac{(\tau-x)^{s-1}e^{\frac{-b}{2\tau}}e^{-(v+1)\tau}}{e^\tau+1}d\tau,$$

which by (5.2) gives (5.24). ∎

Corollary 33 *The following relation is true*

$$2^s\Theta_{\nu+1}(s,x)=\Psi_{\frac{\nu}{2}}(s,2x)-\Psi_{\frac{\nu+1}{2}}(s,2x)$$
$$(\Re(s)>1, x\geq 0, \Re(\nu)>-1). \tag{5.25}$$

Proof. Putting $b=0$ in (5.24) we arrive at (5.25). ∎

Corollary 34 *We have*

$$2^s\Theta_\nu(s,0) = \zeta(s,\frac{\nu+1}{2}) - \zeta(s,\frac{\nu+2}{2})$$
$$(\Re(s) > 1, \Re(\nu) > 0). \tag{5.26}$$

Proof. Taking $x = 0$ in (5.25) and using $\Psi_\nu(s,0) = \zeta(s,\nu+1)$, we obtain

$$2^s\Theta_{\nu+1}(s,0) = \zeta(s,\frac{\nu}{2}+1) - \zeta(s,\frac{\nu+1}{2}+1)$$
$$(\Re(s) > 1, \Re(\nu) > -1).$$

Replacing ν by $\nu - 1$, we arrive at (5.26). ■

Corollary 35 *The eFD and eBE are related by*

$$2^{1-s}\Psi_\nu(s,2x) = \Psi_{2\nu}(s,x) - \Theta_{2\nu}(s,x)$$
$$(x \geq 0, \Re(\nu) > -1, \Re(s) > 1). \tag{5.27}$$

Proof. Putting $b = 0$ in (5.23) we get (5.27). ■

Corollary 36 *The FD and BE functions are related by*

$$2^{1-s}B_{s-1}(2x) = B_{s-1}(x) - F_{s-1}(x) \qquad (\Re(s) > 1, x \leq 0). \tag{5.28}$$

Proof. This follows from (5.27) by setting $\nu = 0$ and replacing x by $-x$. ■

The Fermi–Dirac and Bose–Einstein functions arose in the distribution functions for quantum statistics. They come from the velocity distribution of a quantum gas. Due to their physical significance, these functions have been extensively studied. In this chapter, we have obtained a generalization of the extended Fermi-Dirac and extended Bose-Einstein functions by inserting the factor $e^{-b/t}$, in the integral representations, which plays the role of a regularizer. These generalizations have provided some results of the original Fermi-Dirac and Bose-Einstein functions as well as for other zeta family functions. These functions are related to the family of zeta functions. The family of zeta functions including Riemann, Hurwitz, Lerch, and Hurwitz-Lerch zeta functions have several applications in different areas in applied mathematics and in physics. A generalization of the zeta family is expected to have several useful mathematical and physics applications as well.

CHAPTER 6

CONCLUSION AND FURTHER STUDIES

The Fourier transform representation of the generalized hypergeometric functions was obtained in Chapter 2 to get some integral formulas involving these functions. The Fourier transform representation of the generalized hypergeometric functions led to new results about the confluent and Gauss hypergeometric functions. The main result in that chapter can be used to obtain several formulae by appropriate choices of the parameters.

In Chapter 3, we have represented the generalized hypergeometric functions as a series of Dirac delta functions. This representation has led to some new integral formulae about generalized hypergeometric functions as well as for Gauss and confluent hypergeometric functions. Also, the distributional representation for any Mellin transformable function which has a Laurent or Taylor series has been obtained

$$F_M(\sigma + i\tau) = 2\pi \sum_{n=-\infty}^{\infty} a_n \delta(\tau - i(\sigma + n)) \quad (\sigma \in S_f). \tag{6.1}$$

An application of the distributional representation gave a formula which can be considered as a generalization of Ramanujan's master theorem

$$\frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} F_M(s)\Lambda(s)x^{-s}ds = \sum_{n=0}^{\infty}\frac{f^{(n)}(0)x^n}{n!}\phi(-n).$$

Some applications of the distributional representation are used to find Euler's reflection formula

$$\Gamma(s)\Gamma(1-s) = \frac{\pi}{\sin(\pi s)},$$

and the Riemann functional equation

$$\zeta(s) = 2^s\pi^{s-1}\sin(\frac{\pi}{2}s)\Gamma(1-s)\zeta(1-s).$$

It is expected that the distributional representation (6.1) may be used to find functional equations for other special functions as well. Also, it may help solving some open problems.

In Chapter 4, we have applied Parseval's identity for the Mellin transform to the generalized gamma, the extended beta, the extended Gauss hypergeometric and the extended confluent hypergeometric functions. Several integrals of products involving these extended functions have been obtained. Some applications of Parseval's formula for the incomplete Mellin transform have been discussed. Other special functions having Mellin transform can be used to get more integral formulae.

Generalization of special functions may prove more useful than the original special functions themselves. In Chapter 5, we have obtained a generalization of the extended Fermi-Dirac and extended Bose-Einstein functions by inserting the factor $e^{-b/t}$, in the integral representations, which plays the role of a regularizer. These generalizations functions are related to the family of zeta functions. Applications of these generalizations may lead to some new formulae related to the Riemann zeta function. Some applications are currently under investigation.

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